# MODERN PHYSICS

*for*

# M.Sc. Entrance

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# Chapter 1

# Particle Properties of Wave

## 1.1: Blackbody radiation

A significant hint of the failure of classical physics arose from investigations of thermal radiation (Planck, 1900). According to Einstein (1905) electromagnetic radiation is quantized in photons.

- **Photons and Planck's quantum of action:**

Photons are the energy quanta of the electromagnetic field.

**Photon energy** ($E_{Ph}$) : It is proportional to the frequency f or the angular frequency $\omega = 2\pi f$. Usually it is given in electron volts (eV),

$$E_{Ph} = hf = \hbar\omega$$

**Photon momentum** ($\vec{P}_{Ph}$): It is proportional to the wave number vector k $\left(\text{with } \left|\vec{k}\right|\right) = \frac{2\pi}{\lambda}$, $\lambda$ is the wavelength of the electromagnetic radiation),

$$\vec{P}_{Ph} = \hbar\vec{k}, \qquad \left|\vec{P}_{Ph}\right| = \hbar k = h/\lambda$$

The vector $\vec{k}$ points along the propagation direction of the electromagnetic radiation.

**Planck's quantum of action:** An universal constant,

$$h = 6.626\times10^{-34}\text{ Js},$$

$$\hbar = \frac{h}{2\pi} = 1.0545\times10^{-34}\text{ Js} = 6.5821\times10^{-22}\text{ MeV s.}$$

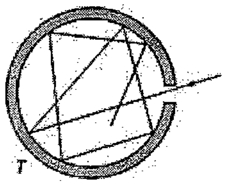
| energy of photon | | | $ML^2T^{-2}$ |
|---|---|---|---|
| | Symbol | Unit | Quantity |
| $E = \hbar\cdot\omega, \quad \omega = 2\pi f,$<br>$\hbar = \frac{h}{2\pi}$ | $E$ | J | energy |
| | $\omega$ | rad $s^{-1}$ | angular frequency |
| | $f$ | $s^{-1}$ | frequency |
| | $h$ | J s | quantum of action |

| momentum of photon | | | $MLT^{-1}$ |
|---|---|---|---|
| | Symbol | Unit | Quantity |
| $\vec{p} = \hbar\cdot\vec{k}, \quad k = \frac{2\pi}{\lambda},$<br>$\hbar = \frac{h}{2\pi}$ | $\vec{k}$ | $m^{-1}$ | wave-number vector |
| | $\hbar$ | J s | quantum of action |
| | $\vec{p}$ | kg m/s | momentum vector |

- **Thermal radiation and the blackbody radiator:**

Thermal radiation, temperature radiation, the electromagnetic radiation of a body at finite temperature. The body also absorbs a fraction of the thermal radiation from its environment. There is a permanent exchange of energy between the body and its environment.

In the end, this process leads to temperature equilibrium. Blackbody radiator, a body with the reflectance zero. A blackbody absorbs any incident radiation completely.



(Figure 2.1: Model of the blackbody)

**Cavity radiator model of a blackbody radiator (Fig. 2.1):** A box with a small aperture in the wall. The wall is impenetrable for radiation from inside (ideally reflecting) and has a definite temperature. The probability that a photon enters the cavity through the aperture and, after multiple reflection by the inner walls, leaves the cavity through the aperture again, is negligible (absorptance $\alpha = 1$). The aperture appears absolutely black.

Cavity radiation, the thermal radiation leaving the aperture of a cavity radiator. The spectral distribution of the radiation energy density of the cavity radiation depends on the temperature of the cavity radiator.

According to Kirchhoff's law (The absorptance is equal to the emittance), the spectral radiance $L_{e,f}$ of an arbitrary thermal radiator may be reduced to that of a black body.

For the radiation field in the interior of the cavity, one defines

| **radiant energy density** | | | **$ML^{-1}T^{-2}$** |
|---|---|---|---|
| | Symbol | Unit | Quantity |
| $u = \dfrac{Q}{V}$ | $u$<br>$Q$<br>$V$ | $J/m^3$<br>$J$<br>$m^3$ | radiant energy density<br>radiant energy<br>volume |

- **Planck's radiation law:**

This law describes the frequency and temperature dependence of the radiant energy density of the cavity radiation:

| **spectral radiant energy density** | | | **$ML^{-1}T^{-1}$** |
|---|---|---|---|
| | Symbol | Unit | Quantity |
| $u_f(f,T) = \dfrac{8\pi f^2}{c^3} \cdot \dfrac{hf}{e^{hf/(kT)} - 1}$ | $u_f(f,T)$<br>$c$<br>$f$<br>$h$<br>$k$<br>$T$ | $J\,s\,m^{-3}$<br>$m\,s^{-1}$<br>$s^{-1}$<br>$J\,s$<br>$J\,K^{-1}$<br>$K$ | spectral radiant energy density<br>speed of light<br>frequency<br>quantum of action<br>Boltzmann constant<br>temperature |

- **Connection between radiant energy density and frequency:**

The dependence of the spectral radiant energy density of the cavity radiation on the angular frequency $\omega$ or wavelength $\lambda$ reads as follows:

$$u_\omega(\omega,T) = u_f(f,T)\frac{df}{d\omega} = \frac{1}{2\pi}u_f(f,T),$$

$$u_\omega(\omega,T)=\frac{\hbar\omega^3}{\pi^2c^3}\frac{1}{e^{\hbar\omega/(kT)}-1},\quad u_\lambda(\lambda,T)=u_f(f,T)\left|\frac{df}{d\lambda}\right|=\frac{f^2}{c}u_f(f,T)$$

$$u_\lambda(\lambda,T)=\frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/(k\lambda T)}-1}$$

- **Limiting cases of Planck's formula:**

**Wien's law:** for $hf >> kT$

$$u_f(f,T)=\frac{8\pi f^3h}{c^3}e^{-\frac{hf}{kT}}$$

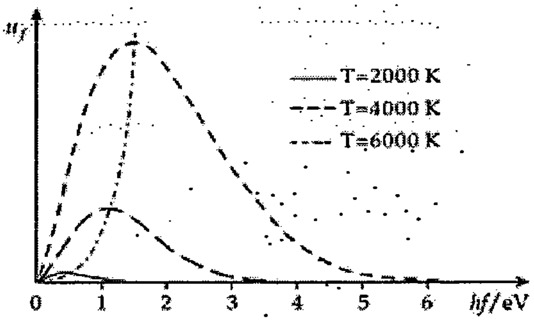
**Rayleigh-Jeans law:** for $hf << kT$

$$u_f(f,T)=\frac{8\pi f^2}{c^3}kT$$

- **Wien's displacement law:**

With increasing temperature, the maximum of the spectral radiant energy density $u_f(f,T)$ is shifted to higher photon energy, i.e., to higher frequencies (shorter wavelengths) (Fig. 2.2)

| Wien's displacement law | | | L |
|---|---|---|---|
| | Symbol | Unit | Quantity |
| $\lambda_{max}=\frac{b}{T}$ | $\lambda_{max}$ | m | wavelength at max. $u_f(f,T)$ |
| $b=2.8978\cdot10^{-3}$ m · K | $b$ | m · K | Wien's constant |
| | $T$ | K | temperature |



(Figure 2.2: Radiant energy density $u_f(f;T)$ for various temperatures according to Planck's radiation law. Dashed-dotted line: Rayleigh-Jeans law.)

- **Stefan-Boltzmann law:**

Integration of the spectral radiant energy density over all frequencies yields the total radiant flux $\varphi_{tot}$ of a radiation emitted by an area A. The total radiant flux $\varphi_{tot}$ is proportional to $T^4$.

| total radiant flux ~ temperature$^4$ | | | $ML^2T^{-3}$ |
|---|---|---|---|
| | Symbol | Unit | Quantity |
| $\Phi_{tot}=\sigma\cdot A\cdot T^4$ | $\Phi_{tot}$ | W | total radiant flux |
| $\sigma=5.67051(19)$ | $A$ | $m^2$ | area |
| $\cdot10^{-8}$ W/(m$^2$K$^4$) | $\sigma$ | W/(m$^2$K$^4$) | Stefan-Boltzmann constant |
| | $T$ | K | temperature |

## 1.2. Photoelectric effect

- **Lenard's experiments**

To study the phenomenon, Lenard set up an apparatus similar to that illustrated in Figure given below. It consists of an evacuated tube $T$ having two plane metal plates $C$ and $D$, $C$ is coated with alkali metal and kept at a negative potential by means of a battery $B$. $A$ is another metal placed parallel to $C$ inside the tube and is kept at zero potential. $A$ is thus positive relative to $C$. At the centre of $A$ there is a small hole through which the beam of negative ions emitted from $C$ can pass. There is a side tube covered by a quartz window $G$ through which UV-radiation is incident on the plate $C$, resulting in the production of negative ions from $C$ photoelectrically. The negative ions are attracted to $A$ and a part of it, passing through the hole in $A$, is finally collected at the collector $D$. The feeble electric current can be detected with an electrometer (not shown).

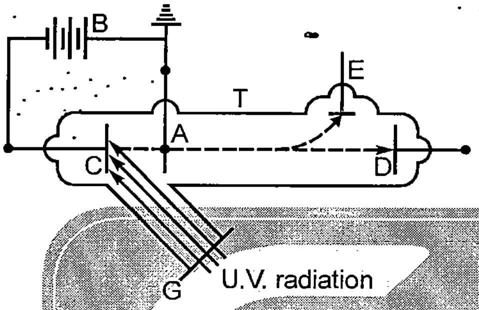


**Fig.** Lenard's experiments on photoelectricity

The main tube is placed between the pole pieces of a magnet such that the field is perpendicular to the plane of the diagram. The field bends the path of the negative ions emanating from the hole into circular arcs so that the ions are now collected by the collector $E$, instead of $D$. Another electrometer (not shown) records the correspondings current.

Let $V$ be the p.d. between $C$ and $A$. Then the kinetic energy of the negative ions is

$$\frac{1}{2}mv^2 = eV \qquad \text{...(i)}$$

If $R$ be the radius of curvature of the ion track, known from the geometry of the apparatus, and $B$ the magnetic field

$$Bev = \frac{mv^2}{R} \qquad \text{...(ii)}$$

From the relations (i) and (ii),

$$\frac{e}{m} = \frac{2V}{B^2R^2} \qquad \text{...(iii)}$$

Lenard's measurements of $e/m$ of the negative ions established uniquely the identify of the ions to be electrons. Lenard next observed that if a positive potential is applied to $C$, the electric current recorded decreases with the increase of positive potential; a stage comes when the current becomes zero. This is because electrons being negatively charged experience a repulsive force in going towards $A$. If $V_s$ be the potential difference between $C$ and $A$ when the current is zero, $V_s$ is called the stopping potential – because electrons of all velocities up to the maximum $v_m$ emitted from $C$ are stopped from reaching $A$. Under this condition

$$\frac{1}{2}mv_m^2 = eV_s$$

$V_s$ is usually of the order of few volts.

Even if the intensity of the incident radiation is now increased, it will cause no electron to reach $A$. However, on making $C$ negative w.r.t. $A$, an increase in the intensity of illumination (the frequency and stopping potential remaining, the same) causes the current to increase.

If the intensity is kept constant, but the frequency of the incident radiation is varied, it is observed that higher the frequency of radiation, greater is the stopping potential and also the energy of the electrons.

Another very important observation that Lenard made during his experiment is that the photoelectric current is independent of the frequency of light. It depends, as already stated, only on the intensity of the light used.

Subsquently, after Lenard, many other workers – notably O.W. Richardson, K.T. Compton – confirmed the observations of Lenard.

**Note 1:** To explain the findings of Lenard on photoelectricity. Einstein in 1905 proposed his celebrated photoelectric equation based on the quantum ideas of Max Planck.

**Note 2:** The most outstanding series of experiments on photoelectricity was however conducted in 1916 by R.A. Millikan, which not only re-confirmed Lenard's observations but also verified beyond doubt the validity of Einstein's equation.

### • Failure of the electromagnetic wave theory

The above experimental facts cannot be explained on the basis of the electromagnetic wave theory of light.

According to the electromagnetic theory, light radiations consist of electric and magnetic fields and they may exert forces to liberate electrons from the metal surface. On this basis, the light of higher intensity, consisting of stronger fields, should give higher velocity (or energy) to the emitted photoelectrons. So, according to the wave theory, the energy of the emitted electron should depend on the intensity of incident radiation which, however, is contrary to experimental observation.

Also, according to the electromagnetic theory, the velocity (or kinetic energy) of electrons should not depend on the frequency of radiation for whatever be the frequency, the electron would be emitted if it gets sufficient time to gather the necessary energy for emission. Further, if the intensity of the radiation be very weak, classically a longer time should elapse before the electrons gain sufficient energy to leave the emitting surface. Both the conclusions are in disagreement with observed facts – the velocity of photoelectron is frequency dependent and that there is no observable between the incidence of radiation and the emission of photoelectron.

Also, according to electromagnetic wave theory, given sufficient time, all electrons should be able to collect the energy required for emission. There is no reason therefore why the photoelectric current should be intensity dependent, as is observed.

The wave theory thus completely fails to explain the characteristics of the photoelectric emission.

### • Einstein's light quantum hypothesis: photoelectric equation

Max Planck, in 1901, put forward a radically new hypothesis to explain the black body radiation that emission and absorption of radiation is not a continuous process but occur discretely as an integral multiple of a basic unit, called the quantum energy. Each quantum carries a definite amount of localised energy $E$ proportional to the frequency $\nu$ of the radiation.

Therefore, $E = h\nu$

where $h$ is a universal constant known as Planck's constant of action. Emission and absorption of energy take place in discrete steps of $nh\nu$, $n = 1, 2, 3, ....$

**Einstein's light quantum hypothesis**—Einstein (1905) proposed the correct explanation of the photoelectric phenomenon by an extension of the quantum idea of Planck. This extension is known as Einstein's light quantum hypothesis.

Einstein proposed the following postulates.

1. Radiation is not only emitted and absorbed in discrete quanta, called photons, having energy $E = h\nu$, but also travels in quanta with the speed of light. Each photon is a bundle of localised energy in space.
2. The energy $h\nu$ of a photon is so localised or concentrated that it can transfer the whole of its energy content to one electron (which thereby gains the quantum of energy) and that no fractional measures are allowed in the process.

In explaining the photoelectric effect, Einstein also postulated that electrons do not normally leave a metal surface of their own because they are bound in the metal. A certain minimum energy, called the work function. $\phi$, is required to release an electron from the metal surface.

## • Photoelectric equation

When photons fall on a metallic surface, they can either (i) be reflected according to the laws of optics or else, (ii) be absorbed completely by an electron in being knocked out from the surface. In the latter case, the emission of electron is possible only when $h\nu > \phi$, and the balance of energy $(h\nu - \phi)$ is taken up by the emitted electron as its kinetic energy in emission. Those electrons which are ejected from some depth below the surface expend some of their acquired energy in collisions with atoms. The energy lost in this way is a variable quantity so that photoelectrons are emitted with a range of kinetic energies (velocities). Thus the maximum kinetic energy of the emitted photoelectrons is given by.

$$\frac{1}{2} m v_m^2 = h\nu - \phi \qquad \text{...(i)}$$

When the full energy of the photon is used up in releasing the electron from the metal and nothing is left as the kinetic energy of the electron,

$$h\nu - \phi = 0$$

Therefore, $$h\nu_0 = \phi \qquad \text{...(ii)}$$

where $\nu_0$ is the corresponding incident frequency and is called the threshold frequency.

Therefore, from (i), $$\frac{1}{2} m v_m^2 = h\nu - h\nu_0 = h(\nu - \nu_0) \qquad \text{...(iii)}$$

Equation (i) and equation (ii) are the two alternative expressions of the well known photoelectric equation of Einstein.

The equations (i) to (iii) above clearly demonstrate the following features.

(i) The maximum kinetic energy of the emitted photoelectrons depends only on the frequency and not on the intensity of the incident light, and maximum kinetic energy increases linearly with the incident frequency.

(ii) If $\nu < \nu_0$, the kinetic energy of the photoelectrons would be negative which is impossible. This means that there is a threshold frequency $\nu_0$, depending on the work function $\phi$ of the particular metal, below which no photoelectric emission occurs.

(iii) The increase in intensity of incident radiation of a given frequency $\nu (\geq \nu_0)$ is equivalent to an increase in the number of photons striking the emitting surface per second, but the energy $h\nu$ of each photon will remain the same. Hence the number of electrons emitted per second, that is, photocurrent will correspondingly increase, but the maximum energy will remain the same.

(iv) However low the intensity may be, whenever a photon with $\nu \geq \nu_0$ falls on the surface of the emitter, emission occurs by the immediate absorption of photon energy by some electron that escapes instantaneously with no observable time lag.

Thus Einstein's photoelectric equation can explain all the photoelectric laws obtained experimentally.

**Note:** Materials in solid, liquid or gasesous state, other than metal surface, emit photoelectrons if irradiated with e.m. radiation of different frequencies from $\gamma$-rays and $x$-rays to ultraviolet and visible radiation.

## • Verification of Einstein's equation: Millikan's experiments

Millikan, in 1914-16, carried out a set of experiments which first verified quantitatively the Einstein's photoelectric equation. These investigations also made the first direct (and accurate) photoelectric determination of Planck's constant $h$ and also the value of work function $\phi$.

**Apparatus** — The principal parts of the apparatus used by Millikan are illustrated in Figure below. B is a small aperture through which light in a narrow frequency range enters the highly evacuated tube $T$ via the quartz window $D$ and falls on the rotation of a wheel $W$ without disturbing the vacuum. By means of the knife $N$ and the rotating wheel $W$, any one of the metal surfaces could be shaved off so that a clean surface always faces the radiation.

The emitted electrons are collected by $C$, a collector grid (electrode) very near to $D$. It is made of oxidised copper wire so that electrons are not emitted from its surface when incident radiation passes through it. An electrometer $E$ connected to the collector measures the photocurrent. Applying a positive potential to the wheel $W$ via $F$, a retarding field (that can be varied by varying the potential) could be set up between $W$ and $C$ to check the less energetic reaching the collector.

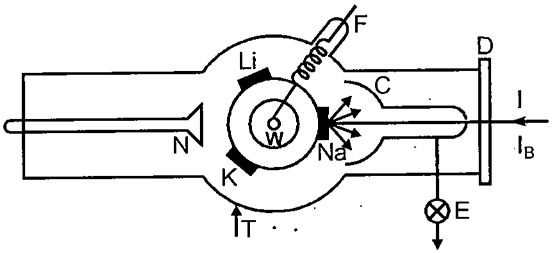


**Fig.** Millikan's experiments on verification of Einstein's equation

**Experiments**—Using monochromatic incident radiations of different wavelengths, Millikan made a careful study of the photoelectric current $i$ for the various retarding potentials $V$ for each of the alkali metals, and the behaviour of current as a function of potentials, keeping the wavelength of radiation fixed. The results of such measurements are shown by the curves in Figure. The different curves are representative of different intensities $I_1, I_2, I_3$ $(I_1 > I_2 > I_3)$. When the potential of $C$ is positive relative to $W$, the current is almost constant (saturated).

**Saturation current**—for a given intensity. The saturation current is reached when all the electrons emitted from the metal surface reached the collector $C$. Interestingly, there is some photoelectric current even when $V = 0$. This shows that some of the electrons must have been emitted with a finite initial velocity.

If the potential of $C$ is made negative relative to $W$, the current decreases and goes to zero at a particular potential $(-V_s)$ which is the same for all intensities. The negative potential $V_s$ is known as the stopping potential and is connected to the maximum kinetic energy by

$$eV_s = \frac{1}{2}mv_m^2 = h(\nu - \nu_0) \quad \text{...(i)}$$

$$\text{or} \quad V_s = h(\nu - \nu_0)/e \quad \text{...(ii)}$$

where $e$ is the electronic charge.

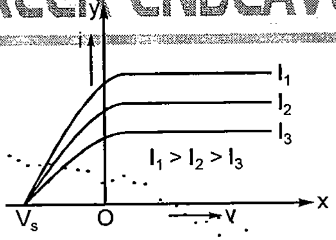


**Fig.** Photoelectric current vs potential curve at different intensities (same frequency) of radiation.

Millikan thus measured the maximum kinetic energy of the electrons emitted by a given frequency (or wavelength). When the intensity of the incident radiation is increased, the saturation current increases (that is, more electrons are ejected), but none of the electrons is more energetic, since the stopping potential remains unchanged. The stopping potential is independent of the intensity of the incident radiation.

Millikan next measured the variation of photocurrent with potential difference between $C$ and $W$ using incident light of different frequencies (or wavelengths), but of the same intensity. The saturation current is the same indicating that number of electrons emitted in each case is the same. But the striking point is that the stopping potential has larger negative value for higher frequency. This is illustrated in Figure where the stopping potential is higher (more negative) as the incident frequency increases $(\nu_3 > \nu_2 > \nu_1)$. But the saturation current is independent of the frequency, being dependent only on the intensity.

Millikan next measured the variation of photocurrent with potential difference between $C$ and $W$ using incident light of different frequencies (or wavelengths), but of the same intensity. The saturation current is the same indicating that number of electrons emitted in each case is the same. But the striking point is that the stopping potential has larger negative value for highest frequency. This is illustrated in figure where the stopping potential is (more negative) as the incident frequency increases $(\nu_3 > \nu_2 > \nu_1)$. But the saturation current is independent of the frequency, being dependent only on the intensity.

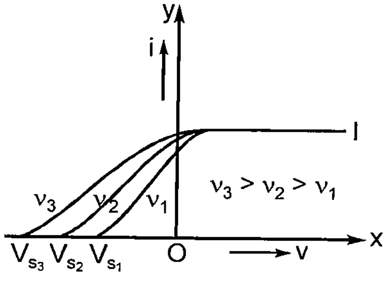


**Fig.** Photo current vs potential curve at different frequencies (same intensity) of radiation

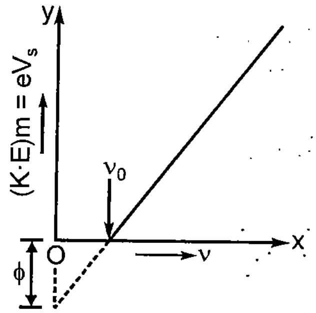


**Fig.** Graph of $(K{\cdot}E)_{\max}$ as a function of ν .

Millikan repeated the experiments by measuring $V_s$ for a range of frequencies and for all the three surfaces. A graph of $(K \cdot E.)_{\max}$ as a function of $\nu$ was obtained figure by Millikan. The graph was a straight line showing that the maximum kinetic energy of the photoelectrons increases linearly with $\nu$. The straight line intercepted the $\nu$-axis at some frequency $\nu_0$. This is the threshold frequency. The equation of the straight line is

$$\frac{1}{2}mv_m^2 = \text{constant} \times (\nu - \nu_0) \qquad \text{...(iii)}$$

Whatever be the metal the slope of the line was always the same, although $\nu_0$ differed from one metal to another. These experimental results were thus is agreement with Einstein's equation.

**Note 1:** While measuring the potential $V$, an error arises due to contact potential difference which was corrected by Millikan by introducing an additional electrode in his apparatus.

**Note 2:** It is to be noted that the Einstein's theory implies that the emission of one electron occurs by the absorption of one photon. But it does not follow that every incident photon causes emission of an electron even if the photon frequency $\nu \geq \nu_0$, the threshold frequency. This is because the photon might be involved in many other alterniative processes to be discussed later. Thus the ratio of the number of emitted electrons and that of the incident photons is much less than unity. This ratio is called the quantum yield and is of importance in the design of photoelectric devices.

## SOLVED PROBLEMS

1. Find the wavelength and frequency of a 1.0 keV photon.

**Soln.** Wavelength of the photon $\lambda = \dfrac{hc}{E} = \dfrac{12.4\ \text{keV}}{1.0\ \text{keV}} = 12.4\ \text{Å}$

Frequency of the photon $\nu = \dfrac{c}{\lambda} = \dfrac{3\times10^{8}\ \text{m/s}}{12.4\times10^{-10}\ \text{m}} = 2.42\times10^{14}\ \text{Hz}$

2. Find the momentum of a 12.0 MeV photon

**Soln.** Momentum of the photon $p = \dfrac{E}{c} = 12\ \text{MeV}/c$

3. Monochromatic light of wavelength 3000Å is incident normally on a surface of area 4 cm$^2$. If the intensity of the light is $15\times10^{-2}$ W/m$^2$, determine the rate at which photons strike the surface.

**Soln.** The energy per photon is

$$E = \frac{hc}{\lambda} = \frac{\left(6.63\times10^{-34}\ \text{Js}\right)\left(3\times10^{8}\ \text{m/s}\right)}{3\times10^{-7}\ \text{m}} = 6.63\times10^{-19}\ \text{J/s}$$

The total energy flux is:

$$IA = \left(15\times10^{-2}\ \text{W/m}^2\right)\left(4\times10^{-4}\ \text{m}^2\right) = 6\times10^{-5}\ \text{W} = 6\times10^{-5}\ \text{J/s}$$

Hence, the rate at which photons strike the surface is:

$$\frac{6\times10^{-5}\ \text{J/s}}{6.63\times10^{-19}\ \text{J/photon}} = 9.05\times10^{13}\ \text{Photon/s}$$

4. The kinetic energies of photoelectrons range from zero at $4.0\times10^{-19}$ J when light of wavelength 3000Å falls on a surface. What is the stopping potential for this light?

**Soln.** Maximum kinetic energy of the photoelectrons $K_{\max} = 4.0\times10^{-19}\ \text{J}\times\dfrac{1\ \text{eV}}{1.6\times10^{-19}\ \text{J}} = 2.5\ \text{eV}$

Then, from $eV_s = K_{\max}$, $V_s = 2.5\ \text{V}$

5. What is the threshold wavelength for above problem?

**Soln.** From the photoelectric equation

$$eV_s = h\nu - eW_0 = \frac{hc}{\lambda} - \frac{hc}{\lambda_{\text{th}}} \quad \text{or} \quad 2.5\ \text{eV} = \frac{12.4\times10^{3}\ \text{eV.Å}}{3000\ \text{Å}} - \frac{12.4\times10^{3}\ \text{eV Å}}{\lambda_{\text{th}}}$$

Solving, $\lambda_{th} = 7590\text{Å}$

6. The emitter in a photoelectric tube has a threshold wavelength of 6000Å. Determine the wavelength of the light incident on the tube if the stopping potential for this light is 2.5 V.

**Soln.** The work function is $W_n = h\nu_{\text{th}} = \dfrac{hc}{\lambda_{\text{th}}} = \dfrac{12.4\times10^{3}\ \text{eV Å}}{6000\ \text{Å}} = 2.07\ \text{eV}$

The photoelectric equation then gives

$$eV_s = h\nu - eW_0 = \frac{hc}{\lambda} - eW_0 \quad \text{or} \quad 2.5\ \text{eV} = \frac{12.4\times10^{3}\ \text{eV Å}}{\lambda} - 2.07\ \text{eV}$$

Solving, $\lambda = 2713\ \text{Å}$

7. The work function of a particular emitter is 2.0 eV and light of wavelength 3000 Å is used to cause emission. Find the stopping potential and the velocity of most energetic electrons.

**Soln.** Light wavelength, $\lambda = 3000\text{ Å} = 3000 \times 10^{-10}\text{m}$.

$$\therefore\ h\nu = \frac{hc}{\lambda} = \frac{6.62 \times 10^{-34} \times 3 \times 10^{8}}{3000 \times 10^{-10}}\text{J} = 6.62 \times 10^{-19}\text{J} = \frac{6.62 \times 10^{-19}}{1.6 \times 10^{-19}}\text{eV} = 4.14\,\text{eV}$$

Now, from the relation $V_s e = h\nu - \phi$, we get the stopping potential, $V_s = 4.14 - 2.0 = 2.14\,\text{V}$.

Again, the kinetic energy of the electron, $\frac{1}{2}mv^2 = V_s e$

$$\therefore\ v = \sqrt{\frac{2V_s e}{m}} = \sqrt{2 \times 2.14 \times 1.76 \times 10^{11}} \qquad (\because\ e/m = 1.76 \times 10^{11}\,\text{C/kg})$$

$$= 8.68 \times 10^{5}\text{ m/s.}$$

8. The wavelength of the photoelectric threshold of tungsten is 2300 Å. Determine the kinetic energy of electrons ejected from the surface by ultraviolet light of wavelegth 1800 Å.

**Soln.** The work function $\phi = h\nu_0 = \frac{hc}{\lambda_0}$

Therefore, $\phi = \frac{6.62 \times 10^{-34} \times 3 \times 10^{8}}{2300 \times 10^{-10}} = 0.86 \times 10^{-18}\,\text{J}$

The energy of the incident radiation is $h\nu = \frac{hc}{\lambda} = \frac{6.62 \times 10^{-34} \times 3 \times 10^{8}}{1800 \times 10^{-10}}\text{J} = 1.10 \times 10^{-18}\,\text{J}$

Therefore, kinetic energy of the ejected electron $E$ is given by

$$E = h\nu - \phi = (1.10 - 0.86) \times 10^{-18}\,\text{J} = \frac{0.24 \times 10^{-18}}{1.6 \times 10^{-19}}\text{eV} = 1.5\,\text{eV}$$

9. The work function of potassium is 1.9 V. If radiation of wavelength $4.5 \times 10^{-7}$ m be incident on it, what would be (i) the threshold wavelength, (ii) the maximum kinetic energy of ejected electrons and (iii) the minimum retarding potential ?

**Soln.** By the problem, $h\nu_0 = \frac{hc}{\lambda_0} = 1.9\,\text{eV}$

Therefore, $\lambda_0 = \frac{hc}{1.9\text{eV}} = \frac{6.62 \times 10^{-34} \times 3 \times 10^{8}}{1.9 \times 1.6 \times 10^{-19}}\text{m} = 6.53 \times 10^{-7}\,\text{m}$

Now, the maximum K·E., $E_m = h\nu - \phi = \frac{hc}{\lambda} - \phi$

Therefore, $E_m = \left\{\frac{6.62 \times 10^{-34} \times 3 \times 10^{8}}{4.5 \times 10^{-7}} - 1.9 \times 1.6 \times 10^{-19}\right\}\text{J} = (4.41 - 3.04) \times 10^{-19}\,\text{J} = 1.37 \times 10^{-19}\,\text{J}$

Minimum retarding potential $V$ is given by $eV = E_m$

Therefore, $V = \frac{E_m}{e} = \frac{1.37 \times 10^{-19}}{1.6 \times 10^{-19}} = 0.86$ volt.

10. A ray of ultraviolet light of wavelength 3000 Å falling on the surface of a material whose work function is 2.28 eV, ejects an electron. What will be the velocity of the emitted electron ?

**Soln.** The work function, $\phi = 2.28\,\text{eV}$

$$= 2.28 \times 1.6 \times 10^{-19}\ \text{J}$$

$$= 3.648 \times 10^{-19}\ \text{J}$$

Wavelength of light, $\lambda = 3000\ \text{Å} = 3000 \times 10^{-10}\,\text{m}$

$$\therefore \text{ Frequency, } \nu = \frac{c}{\lambda} = \frac{3\times10^{8}}{3000\times10^{-10}} = 10^{15}\ Hz$$

Now, kinetic energy of emitted electron is given by $\frac{1}{2}mv^2 = h\nu - \phi$

$$= 6.62\times10^{-34}\times10^{15} - 3.648\times10^{-19}$$

$$= 2.972\times10^{-19}\ J$$

$$\therefore\ v = \sqrt{\frac{2\times2.972\times10^{-19}}{9\times10^{-31}}} = 8.13\times10^{5}\ m/s.$$

11. Calculate the value of photoelectric work function for solution metal in electron volt, given that the threshold wavelength is 6800 Å and $h = 6.62\times10^{-34}\ Js.$

**Soln.** Threshold wavelength $\lambda_0 = 6800\ \text{Å} = 6800\times10^{-10}\,m$

Threshold frequency $\nu_0 = \frac{c}{\lambda_0} = \frac{3\times10^{8}}{6800\times10^{-10}}\ Hz$

Photoelectric work function of sodium

$$\phi = h\nu_0 = \frac{6.62\times10^{-34}\times3\times10^{8}}{6800\times10^{-10}}\ J = \frac{6.62\times10^{-34}\times3\times10^{8}}{6800\times10^{-10}\times1.6\times10^{-19}}\ eV = 1.82\,eV.$$

12. The work function of a metal is $3.31\times10^{-12}$ erg. Calculate the maximum wavelength of radiation which would start the emission of photoelectrons ($h = 6.62\times10^{-27}$ erg. sec, $c = 3\times10^{10}$ cm/s).

**Soln.** Let $\nu_0$ be the threshold frequency of radiation.

Therefore, work function, $\phi = h\nu_0$

$$\text{Therefore, } \nu_0 = \frac{\phi}{h} = \frac{3.31\times10^{-12}}{6.62\times10^{-27}} = 5.0\times10^{14}\,s^{-1}$$

$$\text{Therefore, threshold wavelength, } \lambda_0 = \frac{c}{\nu_0} = \frac{3\times10^{10}}{5\times10^{14}} = 6\times10^{-5}\,cm = 6000\,\text{Å}.$$

13. The maximum energy of photoelectrons emitted by a metal surface is $3.62\times10^{-19}\,J$ when the incident radiation is 3000 Å. When the incident radiation is 5000 Å, the maximum energy of ejected electrons is 0.972 × $10^{-19}$ J. Calculate the Planck's constant and the threshold wavelength of metal.

**Soln.** From the photoelectric equation $(K.E.)_{\max} = h\nu - \phi$ ...(i)

Here, $\nu_1 = \frac{c}{\lambda_1} = \frac{3\times10^{8}}{3000\times10^{-10}} = 1.0\times10^{15}\ Hz$ ; Also, $\nu_2 = \frac{c}{\lambda_2} = \frac{3\times10^{8}}{5000\times10^{-10}} = 0.6\times10^{15}\ Hz$

From equation (i), we obtain

$$\left.\begin{aligned} 3.62\times10^{-19} &= h(1.0\times10^{15})-\phi \\ \text{and }\ 0.972\times10^{-19} &= h(0.6\times10^{15})-\phi \end{aligned}\right\}$$

Solving these two simultaneous equations $h = 6.62\times10^{-34}\,Js$ and $\phi = 3.0\times10^{-19}\,J$.

Threshold wavelength, $\lambda_0 = \dfrac{hc}{\phi} = \dfrac{6.62\times10^{-34}\times3\times10^{8}}{3.0\times10^{-19}} = 6.62\times10^{-7}\,m = 6620\,\text{Å}$.

14. The work functions of tungsten and barium are 4.52 and 2.5 eV respectively. Calculate the wavelength of light which can just eject electrons from tungsten and from barium. Which metal would you select for a photocell for use with visible light ?

**Soln.** Since the work function $\phi = h\nu_0$ where $\nu_0$ is the threshold frequency, $\nu_0 = \phi/h$.

$\therefore$ Threshold wavelength, $\lambda_0 = \dfrac{c}{\nu_0} = \dfrac{ch}{\phi}$ ($c$ = velocity of light)

For tungsten, $\phi = 4.52\,eV = 4.52\times1.6\times10^{-19}\,J$

Therefore, $\lambda_0 = \dfrac{3\times10^{8}\times6.62\times10^{-34}}{4.52\times1.6\times10^{-19}} = 2.746\times10^{-7}\,m = 2746\,\text{Å}$

For barium, $\lambda_0 = \dfrac{4.52}{2.5}\times2746\,\text{Å} = 4965\,\text{Å}$

Plainly, barium would be the metal of choice for a photocell for use with visible light.

15. The work function of aluminium is 4.2 eV. Find the kinetic energy of the fastest and the slowest photoelectrons, the stopping potential and the cut-off wavelength when radiation of 2000 Å falls on an aluminium surface.

**Soln.** From Einstein's photoelectric equation $(K.E.)_{\max} = h\nu - \phi$

Now, $h\nu = h\dfrac{c}{\lambda} = \dfrac{6.62\times10^{-34}\times3\times10^{8}}{2000\times10^{-10}} = 9.93\times10^{-19}\,J = \dfrac{9.93\times10^{-19}}{1.6\times10^{-19}}\,eV = 6.2\,eV$

$\therefore$ $(K.E.)_{\max} = (6.2-4.2)\,eV = 2\,eV$. $\therefore$ Stopping potential = 2 volt

Plainly, $(K.E.)_{\max} = 0$ (zero)

Cut-off frequency (threshold), $\nu_0 = \dfrac{\phi}{h} = \dfrac{4.2\times1.6\times10^{-19}\,J}{6.62\times10^{-34}\,Js} = 1.01\times10^{15}\,Hz$

$\therefore$ Cut-off wavelength, $\lambda_0 = \dfrac{c}{\nu_0} = \dfrac{3\times10^{8}}{1.01\times10^{15}} = 2.97\times10^{-7}\,m = 2970\,A$.

16. Calculate the fractional change in the light intensity of a beam of intensity $10^{-7}$ Js$^{-1}$, if one photon is added per second to the beam, the wavelength of light being 6000 A.

**Soln.** The energy of a photon, $E = h\nu = h\dfrac{c}{\lambda} = \dfrac{6.6\times10^{-34}\times3\times10^{8}}{6000\times10^{-10}} = 3.3\times10^{-19}\,J$

Therefore, the number of photons per sec in the beam is $N = \dfrac{10^{-7}}{3.3\times10^{-19}} = 0.30\times10^{12}$

Therefore, fractional change in the number of photons or the fractional change in intensity on addition of 1 photon per second is

$$\frac{1}{N} = \frac{1}{0.30\times10^{12}} = 3.3\times10^{-12}.$$

17. How many photons of $\lambda = 5\times10^{-7}\,m$ must fall per second on a blackened plate to produce a force of 1 dyne? If the mass of the plate is 1 g and its specific heat $0.42\times10^{3}\,\text{Jkg}^{-1}$, at what rate would its temperature rise ?

**Soln.** Momentum of a photon, $p = h/\lambda$. Let $n$ be the number of photons falling per second.

Therefore, total momentum change per sec of incident photons $= n(h/\lambda)$.

Since all the photons are completely absorbed by the plate, the change in momentum per sec $= n(h/\lambda)$ = force produced on the plate.

$$\therefore\quad n(h/\lambda) = \frac{1}{10^{5}} = 10^{-5} \qquad (\because\ 1\text{ Newton} = 10^{5}\text{ dyne})$$

$$\Rightarrow\ n = \lambda\times10^{-5}/h = \frac{5\times10^{-7}\times10^{-5}}{6.62\times10^{-34}} = 7.55\times10^{21}$$

Energy absorbed by the plate per sec, $Q = nh\nu = \frac{nh\nu}{\lambda}$.

If $\Delta T$ be the rise in temperature per sec, then $Q = \text{mass}\times\text{specific heat}\times\Delta T = \text{ms}\Delta T$

$$\Rightarrow\ \Delta T = \frac{Q}{ms} = \frac{nhc}{\lambda ms}$$

But, we have just seen that $\frac{nh}{\lambda} = 10^{-5}$ $\therefore$ $\Delta T = 10^{-5}c/(10^{-3}\times0.42\times10^{3})$

$$\text{or,}\quad \Delta T = \frac{10^{-5}\times3\times10^{8}}{10^{-3}\times0.42\times10^{3}} = \frac{3\times10^{3}}{0.42} = 7.14\times10^{3}\,K.$$

18. The photon energy for light of wavelength $3.1\times10^{-7}$ m is $\left[hc = 1.24\times10^{-6}\text{ eV m}\right]$ **[BHU 2010]**

(a) 3.1 eV (b) 6.2 eV (c) 2.4 eV (d) 4.0 eV

**Soln.** The photon energy is

$$E = \frac{hc}{\lambda} = \frac{1.24\cdot10^{-6}\text{ eVm}}{3.1\times10^{-7}} = 4\text{ eV}.$$

**Correct option is (d)**

19. The photoelectric threshold for tungsten is 2300 Å. The energy of the electrons emitted from the surface by the ultraviolet light of wavelength 1800 Å incident on the tungsten $\left(h = 6.63\times10^{-34}\text{ J-sec}\right)$ is **[BHU 2011]**

(a) $2.39\times10^{-12}$ joule (b) $2.39\times10^{-19}$ joule (c) $1.48\times10^{-19}$ joule (d) $3.56\times10^{-19}$ joule

**Soln.** Kinetic energy of the emitted electrons will be

$$E_k = \frac{hc}{\lambda} - \frac{hc}{\lambda_{th}} = \frac{1240\,eV-nm}{180\,nm} - \frac{1240\,eV-nm}{230\,nm} = 1.5\,eV = 2.4\times10^{-19}J$$

**Correct option is (b)**

20. The photoelectric equation is derived under the assumption that **[BHU 2012]**

(a) electrons are associated with waves of wavelength $\lambda = \frac{h}{p}$, where $P$ is the momentum

(b) light is emitted only when electrons jump between the orbits

(c) light is absorbed in quanta of energy $E = h\nu$

(d) light behaves like a wave

**Soln.** The photoelectric equation considers the quantization of light, i.e., light is absorbed or removed only in integral multiple of $h\nu$ only.

**Correct option is (c)**

21. The number of electrons emitted increases if the incident light has **[BHU 2012]**

(a) higher frequency (b) higher wavelength (c) higher intensity (d) lower frequency

**Soln.** If the incident light has higher intensity, number of incident photons increases and hence number of emitted electrons also increases.

**Correct option is (c)**

22. The photoelectric threshold for a metal is 3000 Å. The kinetic energy of an electron ejected from it by radiation of wavelength 1200 Å is **[BHU 2012]**

(a) 3.1 eV (b) 12.6 eV (c) 6.2 eV (d) 21.4 eV

**Soln.** The kinetic energy of an ejected electron is

$$\text{K.E.} = h(\nu - \nu_0) = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)$$

$$= 6.6\times10^{-34}\times3\times10^{8}\left(\frac{1}{1200\times10^{-10}} - \frac{1}{3000\times10^{-10}}\right)\text{J}$$

$$= 6.6\times3\times10^{-16}\times\frac{1800}{1200\times3000}$$

$$= 9.9\times10^{-19}\ \text{J}$$

$$\simeq 6.2\ \text{eV}.$$

**Correct option is (c)**

23. A metal surface has work function equivalence to energy of wavelength of light 4500 Å. When a beam of light with wavelength $\lambda$ 4000 Å falls on it **[BHU 2015]**

(a) photoelectrons will not be emitted

(b) photoelectrons will be emitted with certain speeds

(c) photoelectrons will be emitted with zero speeds

(d) photoelectrons may or may not be emitted

**Soln.** Since wavelength of the incident light is less than the threshold wavelength, therefore photoelectrons will be emitted with certain speeds.

**Correct option is (b)**

24. In the phenomenon of the photo electric, the ejected electrons are : **[BHU 2016]**

(a) most of the times relativistic (b) most of the times non-relativistic

(c) always relativistic (d) most of the times stationary

**Soln.** The ejected electrons have kinetic energy of the order of eV as the incident photon has. This makes the electrons non-relativistic in case of photoelectric effect as their rest mass energy is of order of keV.

**Correct option is (b)**

5. Threshold wavelength of a photon that is capable of generating photoelectric effect in a metal, is **[BHU 2016]**

(a) the minimal value (b) the maximum value (c) the average value (d) the root square mean value

**oln.** The threshold wavelength is the maximum wavelength of the incident light to eject photo-electrons from the metals.

**Correct option is (b)**

6. Photons with energies $p\omega$ and $q\omega$ are incident one after another on a metal having work function $\omega$, where $p$ and $q$ are integers. The ratio of respective maximum velocities is **[HCU 2012]**

(a) $\sqrt{\frac{p}{q}}$ (b) $\sqrt{\frac{q}{p}}$ (c) $\sqrt{\frac{p-1}{q-1}}$ (d) $\sqrt{\frac{q-1}{p-1}}$

**oln.** According to Einstein's theory of photoelectric effect,

$$\frac{1}{2}mv_{\max}^2 = h\nu - \phi$$

$$\therefore \frac{1}{2}mv_{\max}^2(p) = p\omega - \phi = p\omega - \omega = \omega(p-1) \Rightarrow v_{\max}(p) = \sqrt{\frac{2\omega}{m}(p-1)}$$

Similarly $v_{\max}(q) = \sqrt{\frac{2\omega}{m}(q-1)}$

$$\therefore \frac{v_{\max}(p)}{v_{\max}(q)} = \sqrt{\frac{p-1}{q-1}}$$

**Correct option is (c)**

7. The threshold wavelength of potassium is 558 nm. If a light of wavelength 400 nm is incident on it the stopping potential will be given by **[HCU 2013]**

(a) 8.88 V (b) 0.88 V (c) 88.8 V (d) 2.22 V

**Soln.** The stopping potential, $V_s = \frac{hc}{e}\left(\frac{1}{\lambda} - \frac{1}{\lambda_T}\right) = \frac{6.62\times10^{-34}\times3\times10^{8}}{1.6\times10^{-19}}\times\left(\frac{1}{400} - \frac{1}{558}\right)\times10^{9}$

$$= \frac{6.62\times3\times158\times10^{-34}\times10^{8}\times10^{9}}{1.6\times10^{-19}\times400\times558} = 8.78\times10^{-1}\text{ V} = 0.88\text{ Volt.}$$

**Correct option is (b)**

28. In the photoelectric effect experiment, if we increase the intensity of the incident light, then **[HCU 2016]**

(a) The kinetic energy of the emitted electrons increase

(b) The kinetic energy of the emitted electrons decreases because the number of electrons emitted increases

(c) The kinetic energy of the emitted electrons does not change but the number of electrons emitted increases

(d) both the kinetics energy and the number of electrons emitted increase

**Soln.** According to quantum theory of light, intensity of light is proportional number of photons. i.e $I \propto n$.

Since the number of emitted electrons is proportional to the number of incident photon, so number of emitted electrons will increase as intensity increases.

But the kinetic energy of the emitted electrons depend only on the frequency of incident photon.

So kinetic energy will not change with the intensity of light.

**Correct option is (c)**

## PRACTICE SET

1. Find the wavelength and frequency of a 1 MeV photon. $\left[1.24\times10^{-2}\,Å;\,2.42\times10^{20}\ \text{Hz}\right]$

2. Find the wavelength and frequency of a photon whose momentum is 0.02 Me V/c.
$\left[6.20\times10^{-1}Å;\ 4.84\times10^{18}\times10^{18}\ \text{Hz}\right]$

3. Find the momentum of a 4 keV photon. [4 KeV/c]

4. Find the energy of a photon whose momentum is 10 MeV/c. [10 MeV]

5. Find the energy of a photon whose wavelength is 4000 Å. [3.1 eV]

6. Find the energy and momentum of a photon whose frequency is $10^6$ Hz.
$\left[4.14\times10^{3}\ \text{MeV};\,4.14\times10^{3}\ \text{MeV/c}\right]$

7. A 300 MHz plane electromagnetic wave is incident normally on a surface of area 50 $cm^2$. If the intensity of the wave is $9\times10^{-5}$ W/$m^2$, determine the rate at which photons strike the surface.
[$2.26\times10^{18}$ photons/s]

8. The maximum energy of the emitted electrons when a material is illuminated with 3000 Å light is 1.2 eV. Find the work function. [2.93 eV]

9. Light of wavelength 4500Å is incident on two photoelectric tubes. The emitter in the first tube has a threshold wavelength of 6000Å and the emitter in the second tube has a work function twice as large as in the first tube. Find the stopping potential in each of the tubes.
[$V_{s_1}=0.69$ V; There is no photoelectric emission from the second tube]

10. Find the strength of the traverse magnetic field required to bend all the photoelectrons within a circle of radius 20 cm when light of wavelength 4000Å is incident on a barium emitter. The work function of barium is 2.5 eV.
[$1.32\times10^{-5}$ T]

11. A 100 keV photon scatters from a free electron initially at rest. Find the recoil velocity of the electron if the photon scattering angle is 180°. (Use energy and momentum conservation) [0.319c]

12. In Compton scattering the scattered photon and electron are detected. It is found that the electron has a kinetic energy of 75 keV and photon an energy of 200 keV. What was the initial wavelength of the photon?
[0.045 Å]

13. Calculate the maximum energy, in electron-volts, that can be transferred to an electron in a Compton experiment when the incident quanta are X-rays of wavelength 0.50 Å. [4.7 eV]

14. An electron that undergoes a "head-on" collision with an X-ray photon has a stopping potential of 70 kV. If the electron was initially at rest, what are wavelengths of the initial and scattered X-ray photons?
[0.0716Å, 0.1201 Å]

# Chapter 2

# X-rays and Compton Scattering

## 2.1. X-rays

### • Introduction

X-radiations form a part of the electromagnetic spectrum and were discovered accidentally by Wilhelm Conrad Röntgen in 1895. While carrying out investigations on the discharge of electricity through rarefied gases, he had observed that the portion of the glass of a cathode ray tube hit by the cathode rays emitted only only fluorescent light but also an invisible radiation along with it. As the nature of the new radiation could not be deciphered immediately, it was termed for the time being, the unknown of X-radiation. The name, however, survives till today although many call it ' Röntgen rays' after its discover.

The discovery of X-rays opened up a new era in physics, a period of unusual growth and unprecedented activity. Not only physics, the other branches of knowledge that reaped immediate benefit from the discovery are medicine, geology and industry. Within a year of the first announcement of the discovery, as many as 49 books and pamphlets had been published on the subject, and more than 1000 articles had appeared in various journals. In commemorating the 50th anniversary of the discovery G.E.M. Jauncey rightly observed : 'November 8, 1895 may be taken as the date of a new era in physics. Before this notable date, it may be said people were living in the classical era.... clinging for dear life to the classical ideas of the existance of luminiferous ether and the non-existence of anything smaller in mass than the hydrogen atom or ion. A great jolt was needed to shake the minds of physicists loose from these ideas.

### • Properties of X-rays

As a result of the work of Röntgen and others, the wave nature and various other properties of the new radiation were subsequently established. The principal properties are :

1. The X-rays, like light, are electromagnetic radiations of extremely short wavelength (100-0.0001 Å). They show reflection, refraction, interference, diffraction and also polarization under suitable conditions.
2. They are unaffected by electric and magnetic fields, indicating that they are not beams of charged particles unlike cathode rays. They travel in straight lines with the speed of light.
3. They posses extremely high penetrating power. The shorter the wavelength, the greater is the power of penetration. They penetrate more in low density substances and less in high density ones. For instances, while ordinary glass is quite transparent to X-rays, lead glass is almost opaque. Usually they pass easily through matter that are opaque to visible rays. Long wavelength X-rays are absorbed by materials to varying degrees and this forms the basis of radiography.
4. They affect (i.e. blacken) photographic plate even when wrapped in black papers by several folds. The degree of blackening depends on the intensity of the radiation. Photography, in fact, is a very valuable technique for the study of X-radiation.
5. They cause fluorescence in many substances such as barium platinocyanide, zinc-sulphide, calcium salts and almost all the organic substances. The fluorescence is such a distinctive characteristic of the substance that it could be easily identical by it.

6. They ionise gases through which they pass, and the ionising power is dependent on and is a measure of the intensity of X-rays. The intensity of X-rays could thus be measured by their ionising power.
7. X-rays also produce photoelectric effect and thus exhibit their cospuscular nature. This nature is further confirmed by the Compton effect of X-radiation.
8. They are also capable of destroying living cells and cause genetic mutation.

## • Application of X-rays

X-radiation find many important and useful applications in diverse fields, e.g. pure science, industry, medicine etc. as has been stated already.

**In science**—X-ray spectroscopy gives a powerful method for identification of chemical elements and determination of atomic structure. X-rays crystallography enables us to realise that all solids – from rock salt to nucleic acid – are crystalline in their natural state, and that almost every aspect of solid behaviour depends on the arrangement of its atoms and molecules. X-rays studies help to detect this arrangement and explain the associated properties of the solids. Such studies are almost a routine affair with geologists and solid state physicists.

**In industry**—X-ray radiography has been utilised in solving problems of the constitution and physical behaviour of metals and alloys, textile fibres, wool, silk, rayon, waxes, soaps, cement, ceramics, sugar, paints, plastics and many other materials. They are used for detecting flaws in casts, invisible surface cracks, defects in welding and in industrial materials such as machine tools and high speed automobile parts, presence of foreign materials etc. by non-invasive (non-destructive) testing. They are also used for fluoroscopic examination of food materials, oil painting and art treasures. The detective department and the customs utilise the radiation to examine boxes, suitcases and such other containers without opening them for contraband goods or dangeraous bombs and explosives inside them.

**In medicine**—In medicine, X-rays are used for both diagnostic and therapeutic purposes. Medical radiography detects and also locates internal defects of human bodies, fractures of bones, presence of such foreign bodies as bullets, stones etc. They are used in diagnosis of tumours, tubercular lungs and bones. Ultrasonography has of late, comp up as a powerful competitor in the field. In the field of therapy, deep X-radiation is used in the treatment of cancer using the property of its selective action in destroying abnormal cells. Fixation of proper dose, and the administration by dosimetry, is extremely important as the normal tissue is also susceptible of being affected by irradiation.

The above enumeration, however, does not exhaust the evergrowing applications of X-rays in almost every field of human activity.

## • Production of X-rays

The principle of the production of X-rays is that they are produced whenever electrons with energies in the region of kV strike heavy metallic targets. The process may thus be considered as the inverse of photoelectric effect. The impinging electrons lose their energy on collision with the target and at each collision a part, or extremely rarely, the whole of the lost energy appears as X-rays.

**Modern X-rays tube**—After his initial experiments, Röntgen designed and developed a new type of discharge tube – gas tube – for the production of X-rays. Such X-ray tubes are now of historical interest only and would not be described here.

One of the modern forms of X-ray tubes is the Coolidge tube (Figure-1) designed and developed by the American physicist W. D. Coolidge in 1913.

It consists essentially of a bulb *B* with two coaxial projecting tubes *T*, *T* in one of which there is a sprial (filament) *F* of tungsten wire which acts as the cathode. It could be heated to a temperature as high as 2000°C by electric current derived from a low voltage battery (L.T.) or a transformer when it would emit thermoelectrons. A metallic cup *C* of molybdenum surrounds the filament *F* in order to focus the thermoelectrons onto the target. The anode *A* is made of copper with a target of tungsten or molybdenum housed in the other projecting tube. The vacuum of the tube is of the order of $10^{-6}$ mm of mercury. A high potential, ~ 50 – 200 kV, is maintained between the cathode and anode by the high tension source H.T. This large potential difference across the tube causes the electrons from the heated cathode to impinge onto the target with terrific kinetic energy and it is this impact that produces X-rays. The intensity of the beam is controlled by resulting the filament current.

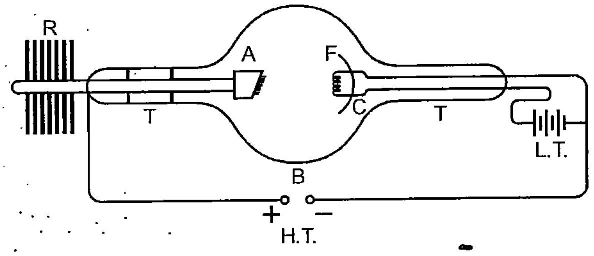


**Fig. 1** Coolidge tube

The impact of the electrons heats up the anode and to prevent melting of anode by excessive heating there is an arrangement of cooling it either by a number of radiator fins *R*, as in small tubes, or by circulating water or oil through the anode which is usually tubular.

The Coolidge tube, unlike gas-filled tubes of Röntgen days, has many advantages. The intensity of X-rays depends, (i) on the current and (ii) the quality of anode potential both of which could be separately controlled in a Coolidge tube; the former by the heating current through the filament and the latter by the applied potential difference. Further, the Coolidge tube is, in general, more stable than the gas tube during continued operation.

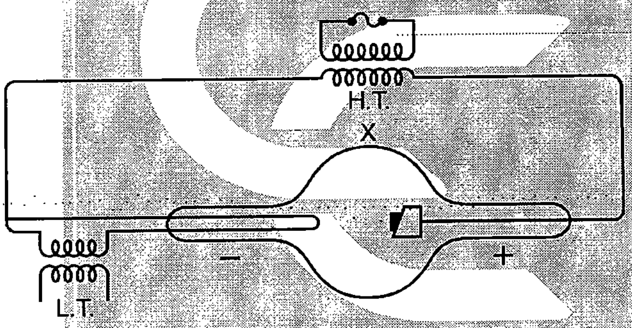


**Fig. 2** Small X-ray plant

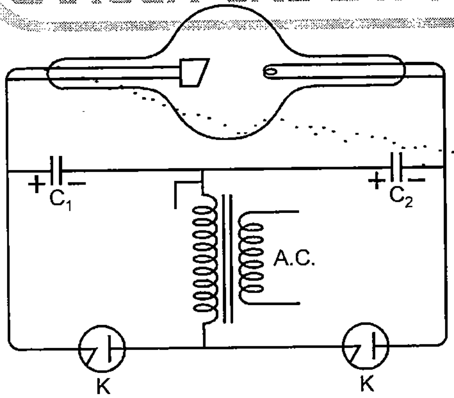


**Fig. 3** Large X-ray plant

**X-ray installation** — In small X-ray plants, where the applied potential difference is realtively small, the cathode is heated by the low tension transformer L.T. (Figure-2) while the accelerating voltage is supplied by high tension transformer H.T. Electrons are attracted towards the anode only during one half of the cycle when the anode is positive. The X-rays are generated during this half cycle. In the other half of the cycle, however, X-rays cannot be produced. The tube thus acts as its own rectifier and suffers from the disadvantage that X-rays are produced only in one half of the cycle thereby shortening the useful life of the bulb.

In large X-ray plants where the applied p.d. is relatively higher, kenotron rectifiers are used. A kenotron is a high voltage diode rectifier. The arrangement for full wave rectification, using two kenotrons *K*, *K*, is shown in (Figure-3). In one half of the cycle, the left kenotron conducts when the condenser $C_1$ gets charged; in the other half, it is the right kenotron that conducts and the condenser $C_2$ is charged. Such a circuit is known as a voltage doubler circuit. This is because the total voltage on the anode is nearly double the voltage of the secondary of the transformer.

**Betaron** — One of the most modern X-ray generators is the betatron devised in 1941 by the American Physicist D.W. Kerst. Here the electrons can be accelerated to a very high energy, ~ $10^8$ eV, and consequently X-rays of extremely high energy are produced. Such high energy X-rays are being fruitfully used in investigating the structure of atomic nuclei apart from then increasing utility in the field of medicine and surgery for both diagnostic and therapeutic purpose.

The construction, principle of action and operation of the betatron has been discussed in the Chapter : Particle Accelerators.

## • Intensity of X-rays: measurement

The measurement of intensity of X-radiation is generally made by ionisation chamber. The principle of the method is as follows: A suitable gas or vapour is taken inside the chamber and the passage of X-rays through it produces ionisation. The ion-current developed is proportional to the X-ray intensity and is measured by an electrometer.

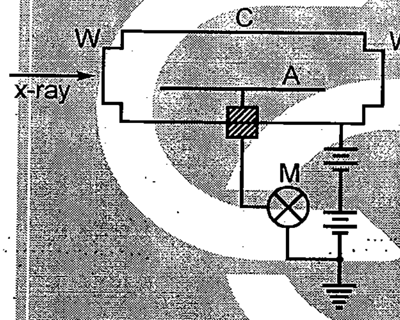


Fig. 4 Measurement of intensity of X-rays

The method is illustrated in Figure-4. *C* is a cylindrical metallic chamber, usually of brass, 0.2 to 1 m long, with the open ends closed by two thin aluminium sheets W, W, known as the windows. X-radiations enter the cylinder through one of the windows and leave it by the other. The chamber is filled with air, ethyl bromide vapour etc. The X-ray beam ionises the gas or vapour inside the chamber. The positive ions are drawn to the tube wall which acts as the cathode and the negative ions move to the anode A, a metallic rod placed inside the chamber parallel to its axis but insulated from the wall. The anode A is connected to an electrometer M and a potential difference of several hundred volts is maintained between A and C. The electrometer M is capable of measuring very small ionisation current produced by X-radiation. The ion-current is found to be linearly related to the X-ray intensity, and thus gives the intensity of the beam.

A scintillation counter (detailed discussion of which is made in Chapter : Nuclear Radiation, Detectors) is especially suitable for measuring the intensity of X-rays from high voltage sources. The X-ray beam is allowed to penetrate a crystal or a liquid and is transformed into the visible fluorescent radiation. This radiation actuates a photomultiplier tube and the resulting photoelectric current is multiplied and measured. The intensity of X-rays is directly proportional to the photocurrent.

## • Variation of X-ray intensity with wavelength

X-rays of different wavelengths are emitted from the X-rays tube and if the intensity is plotted against wavelength, then curves of the nature as shown in Figure-5 and Figure-6 are obtained. The different curves in each Figure corresponds to different voltages used. While Figure-5 gives $I$ vs $\lambda$ graph for a target material of high atomic number, e.g. tungsten ($Z = 74$), Figure-6 gives $I$-$\lambda$ curve when a target material of low $Z$-value, e.g. Mo ($Z = 42$), is used.

The main features of the intensity distribution may be summed up as follows :

1. The intensity increases with increasing potential difference between the cathode and the anode.
2. For the same potential difference the curve, starting from the long wavelength side, attains a maximum and then falls rapidly to zero at the sharp cut-off wavelength $\lambda_0$ or $\lambda_{\min}$ which gets shifted to shorter wavelengths with increasing voltages.
3. For a given wavelength, the intensity is higher, higher the potential difference and as shown in Figure-5 $I$ varies continuously with $\lambda$ up to a voltage difference ~ 50 kV.
4. For the same p.d., the intensity increases with increase of the at. no. $Z$ of the target.

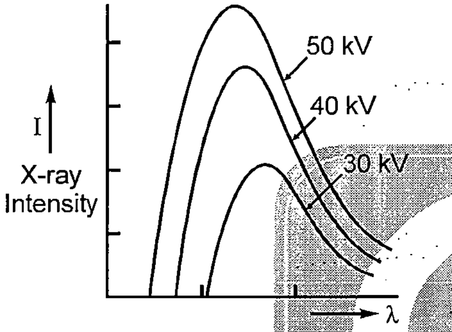


**Fig.5**-Intensity vs. wavelength curves (high Z-value)

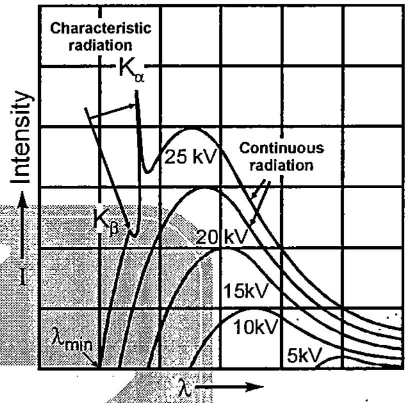


**Fig.6**-Intensity vs. wavelength curves (low Z-value)

5. At potentials ~ 25 kV or higher, several discrete peaks appear, superimposed on the continuous background, at specific wavelengths with target of $Z = 42$ (Figure-6). For target with still lower Z, e.g. Cu ($Z = 29$), the peaks begin to appear at still lower voltage ($V \geq 8$ kV), the peak wavelengths for Cu are however different from those of Mo. The heights of the peaks increase with increasing voltage.
6. The wavelengths of the peaks in the intensity distribution curve are characteristic of the target material and are independent of the applied voltage. These are called characteristic X-rays. Higher the Z, shorter is the wavelength of characteristic X-rays.

**• Origin of continuous X-radiation**

The origin of continuous X-radiation, the existence of minimum wavelength and the calculation of its magnitude can only, and easily, be understood from the quantum concept of radiation.

According to the quantum theory, each X-ray photon is produced as a result of particular collision of one electron. The photon of maximum energy is generated when the electron loses the whole of its energy in a single collision. The electrons are accelerated to an energy $Ve$ by the accelerating voltages $V$. So the maximum energy of the photon is given by

$$E_m = h\nu_m = \frac{1}{2}mv^2 = eV \qquad \text{...(i)}$$

where $\nu_m$ is the maximum frequency of the photon.

But the maximum frequency $\nu_m$ corresponds to the minimum wavelength $\lambda_0$, given by

$$\lambda_0 = \frac{c}{\nu_m} = \frac{hc}{Ve} \qquad \text{...(ii)}$$

where $c$ is the velocity of light in free space.

If $V$ is measured in volt, we have $\lambda_0 = \frac{12413}{V(\text{volt})}\text{Å}$ ...(iii)

So, if $V$= 10,000 volts, $\lambda_0 = 1.24\,\text{Å}$; if $V = 50\,\text{kV}$, $\lambda_0 = 0.248\,\text{Å}$, etc.

The magnitude of $\lambda_0$ can be measured accurately and the equation (ii) may be exploited to find the value of $h$, the Planck's constant. The value of $h$ thus determined is in good agreement with those obtained from thermal radiation and photoelectric effect. Further, equation (iii) or (ii) shows that higher the $V$, lower is $\lambda_0$ ; this is also observed experimentally.

Events in which electron gives up the whole of its content are extremely rare and, in general, an electron produces several photons of wavelength $\lambda > \lambda_0$ in a succession of collisions. The electrons also cause some ionisation and much (sometimes 99%) of the kinetic energy gets converted into heat energy.

**Note 1.** The above mechanism of emission of electromagnetic radiation from an accelerated or decelerated electron is called bremsstrahlung process and the emitted radiation bremsstrahlung (brems = brake, strahlung = radiation, bremsstrahlung = brake-radiation i.e. radiation due to deceleration).

**Note 2.** X-rays are often classified on the basis of their penetrating power. Most penetrating radiation is called hard X-radiation, the least penetrating one the soft X-rays, the intermediate one the medium radiation.

## 2.8. Origin of characteristic X-ray lines

In addition to the smooth continuous spectrum just discussed there appear, as we have seen, some sharp, characteristic X-ray lines at sufficiently high voltages. The mechanism of their production is quite different and lies in the interior electronic structure of the atoms.

The wavelength of X-ray lines are usually of the order of few angstrom or even shorter, implying that their energies are ~ $10^4$ eV or higher. Consequently, the atomic electrons whose transitions give rise to X-ray lines must have binding energies ~ $10^4$ eV or higher. This is possible only if inner atomic orbits close to the nucleus house the electrons.

According to Bohr, an atom of an element of atomic number Z contains in its normal state Z electrons revolving round the nucleus in different shells. The innermost shell corresponds to the first quantum orbit $n = 1$ and is called the K-shell by Kossel. The two electrons that can exist in this shell being closest, to the nucleus require the greatest excitation energy for their removal. Next, in sequence, comes the L-shell with $n = 2$, M-shell with $n = 3$, N-shell with $n = 4$ and so on having electrons that require decreasing excitation energy values for their removal.

Let a target-atom be bombarded by high speed electrons and a K-electron which is tightly bound be knocked out. If $E_K$ be the binding energy of K-electron, then the kinetic energy of the knocked out electron is

$$\frac{1}{2}mv^2 = Ve - E_K$$

where $Ve$ represents the kinetic energy of the incident high speed electron.

So the minimum energy of the incident electron to knock out a K-electron is $E_K$.

Now, on knocking out a K-electron, a vacancy is created in the lowest-lying K-shell and transition of an electron from the higher (and relatively loosely bound) L-shell fills it up. The energy difference $E_K - E_L$, due to the transition results in the emission of $K_\alpha$ -line of frequency $\nu_{KL}$, given by

$$h\nu_{KL} = E_K - E_L$$

So the frequency of $K_\alpha$ -line (or its wavelength) is a characteristic of the target atom. Since the targets used in X-rays tubes are usually of high Z-values, the low-lying levels have binding energies ~ keV. The emitted radiation is thus in keV region, and not in optical region.

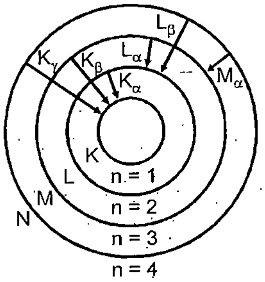


**Fig.7**-Kossel diagram
representing x-radiation series

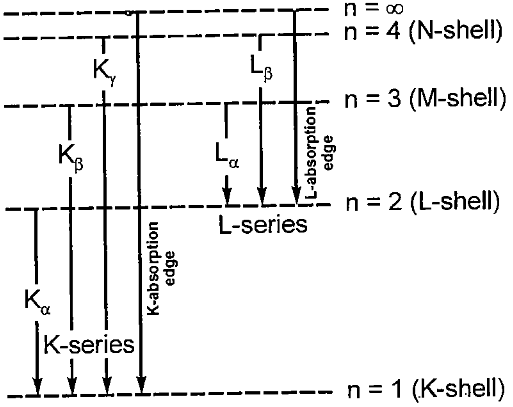


**Fig.8**-Energy level diagram
of x-ray series spectra

If however an electron from the M-shell fills up the said vacancy, the $K_\beta$ -line is emitted and so on, forming the K-series lines. When an electron from the L-shell is knocked out (or falls into the K-shell), electrons froms the M, N,.... shells may fall into the L-shell to fill up the vacancy and the L-series lines are emitted. Other series lines also have a similar origin. The short wavelength limit for each series is called the corresponding absorption edge.

The origin of the lines of different X-radiation series by orbital transitions is best understood and represented by what is called Kossel diagram as shown in Figure-7. The corresponding energy level diagram is presented in Figure-8.

Summing up:

$K_\alpha$ -line : Transition from $K \to L$ level
$K_\beta$ -line : Transition from $K \to M$ level
$L_\alpha$ -line : Transition from $L \to M$ level
$L_\beta$ -line : Transition from $L \to N$ level

The corresponding frequencies of the lines are

$$\nu_{K_\alpha} = \frac{(E_K - E_L)}{h}; \quad \nu_{K_\beta} = \frac{(E_K - E_M)}{h}$$

$$\nu_{L_\alpha} = \frac{(E_L - E_M)}{h}; \quad \nu_{L_\beta} = \frac{(E_L - E_N)}{h}$$

**Note 1:** The above characteristic radiation may as well be generated by allowing the continuous X-radiation to be scattered from different elements outside the X-ray tube. These X-ray photons can knock out K-electrons, L-electrons etc. of the scatterer if the energy of the X-ray photons $h\nu$ is greater than, $E_K$, $E_L$ etc. The excess energy is carried away by the characteristic X-ray photon.

**Note 2:** Since $E_K > E_L > E_M$ etc. it is obvious that $\nu_K > \nu_L > \nu_M ....$ or $\lambda_K < \lambda_L < \lambda_M ....$

## 2.9. Fine structure of characteristic X-ray lines

On careful exmination, it is seen that the characteristic X-ray lines are not single but possess fine structure, i.e. each line consists of a number of closely-spaced components. The origin of this fine structure of X-ray lines can be explained on the basis of quantum numbers $n$, $l$ and $j$ associated with each energy level of the electron. For a given $n$, which determines the electron shell (i.e. $n = 1$ for K-shell, $n = 2$ for L-shell etc.), $l$ can assume values $l = 0, 1, 2, ....(n-1)$. For a given $l$ again, the total quantum number $j$ can have only two values :

$j = l + \frac{1}{2}$ or $j = l - \frac{1}{2}$. For $l = 0$, however, $j = \frac{1}{2}$ only.

The multiplicity of the level corresponding to different $n, l, j$'s are given below in a tabular form.

| Shell | $n$ | $l$ | $j$ | Energy level | Multiplicity |
|---|---|---|---|---|---|
| $K$ | 1 | 0 | $\frac{1}{2}$ | $K_1$ | 1 |
| $L$ | 2 | 0 | $\frac{1}{2}$ | $L_1$ | 3 |
|  |  | 1 | $\frac{1}{2}, \frac{3}{2}$ | $L_{II}, L_{III}$ |  |
| $M$ | 3 | 0 | $\frac{1}{2}$ | $M_1$ | 5 |
|  |  | 1 | $\frac{1}{2}, \frac{3}{2}$ | $M_{II}, M_{III}$ |  |
|  |  | 2 | $\frac{3}{2}, \frac{5}{2}$ | $M_{IV}, M_V$ |  |

Again, the transition between the different levels are determined by certain selection rules.

These are : (i) $\Delta l = \pm 1$ and (ii) $\Delta j = 0, \pm 1$

The fine structure of X-ray levels and the allowed transitions between them are shown in Figure-9. It is observed that each K-line is a doublet : $K_{\alpha_1}$ and $K_{\alpha_2}$, $K_{\beta_1}$ and $K_{\beta_2}$. While $K_{\alpha_1}$ is due to transition from $K \to L_{III}$, $K_{\alpha_2}$ is due to transition from $K \to L_{II}$. The splitting of L.M. etc. lines are relatively more complex.

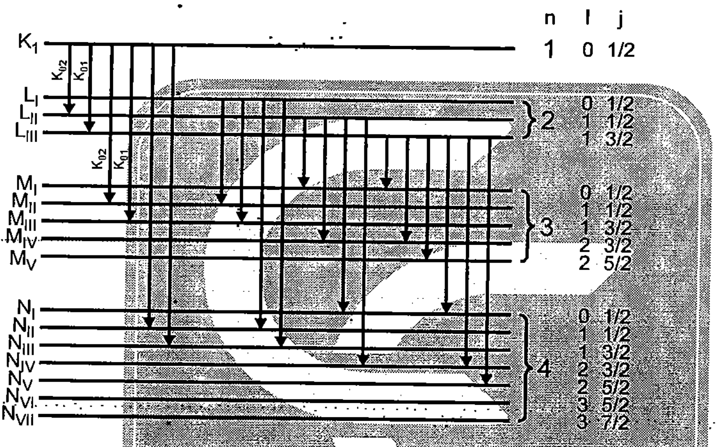


Fig. 9 Fine structure of X-ray levels

## 2.11. Moseley's work on X-ray spectra

A comprehensive investigation of the characteristic X-ray spectra was made by the English Physicist H.G.J. Moseley (1913-14) who used nearly 40 elements from aluminium to gold as targets in the X-ray tube. The radiation from each target was analysed by the single crystal X-ray spectrograph using a thin crystal of potassium ferrocyanide. The spectrum was recorded on a photographic plate and the entire arrangement was placed in a vacuum chamber in order to avoid the absorption of long wavelength X-rays in air.

The photographic plate, on development, showed traces of dark lines of varying intensities, each corresponding to a characteristic line of the given element, The characteristic spectra could easily be divided into two distinct groups – a short wavelength group, generally less than 1 nm (K-series) and a long wavelength group, greater than 1 nm (L-series). The wavelengths of the L-series were roughly ten times as great as those of the K-series. Subsequent investigations confirmed the existence of two other series lines (M-series and N-series) of longer wavelength in heavier atomic species (Z > 66).

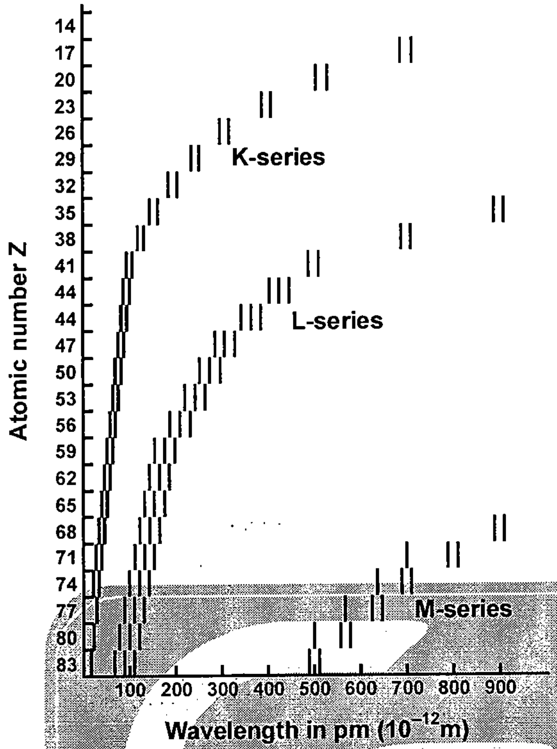


Fig.10-K, L and M-series spectra of different elements

Moseley demonstrated that, unlike the optical spectra, the characteristic X-ray spectra are much simpler in character consisting, as they are, of comparatively fewer lines. Further, they are the same type irrespective of the elements, except that the wavelengths of the corresponding lines decrease with increasing atomic weight. The K, L and M-series spectra of different elements are shown in Figure-10.

**Moseley's law**—Moseley from his investigation deduced the following relation between the frequency $\nu$ of any particular line emitted by an element and its atomic number Z.

$\sqrt{\nu} = a(Z - b)$ where $a$, $b$ are constants characteristic of the element.

This is known as the Moseley's law. By plotting the square root of the frequency of the line (say, the most intense of the K-series) against the atomic number Z of the element concerned, Moseley obtained (Figure-11) a straight line given by the above equation. The same linear relation was found to hold good for any line in any series, of course with different slopes. From Moseley's diagram, $b$ is the small intercept on the Z-axis.

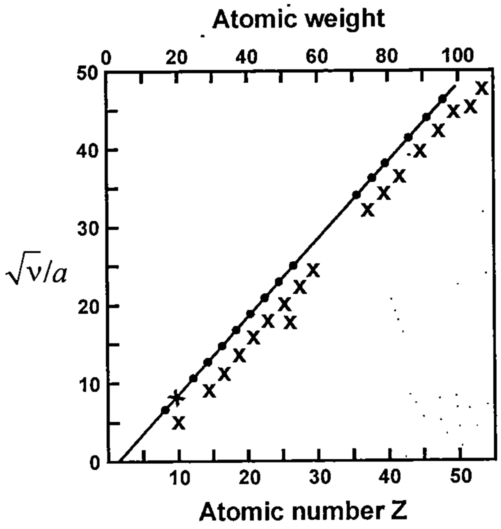


**Fig. 11** Moseley diagram : $\sqrt{\nu}/a$ vs. Z-plot (also at. wt. plot)

In Figure-11 ‘crosses’ correspond to $\sqrt{\nu}/a$ vs atomic weight, while the ‘dots’ correspond to $\sqrt{\nu}/a$ vs atomic number Z. It is apparent from the plot that in so far as the frequency of the characteristic line is concerned, the atomic number is more fundametal than the atomic weight. That is to say, the chemical identify of an element depends on the atomic number Z and not on the atomic weight or mass number A.

**Importance of Moseley’s work**— It was Moseley’s investigation which for the first time showed that it would be more logical to arrange elements in the periodic table in the order on increasing atomic number instead of atomic weight. When this was done, it removed certain anomalies in the order of some elements in the old Periodic table of Mendeleev. For instance, argon of atomic weight 40 comes before potassium of atomic weight 39 from the consideration of chemical properties of these two elements. The atomic numbers of argon and potassium are however 18 and 19 respectively. So the arrangement is correct if made in the order of atomic number. Other such examples are cobalt nickel and tellurium iodine.

Moseley’s work also helped (i) in the discovery of new elements in the missing gaps of atomic number 43, 61, 72 and 75 in the Periodic table and (ii) in the determination of atomic numbers of rare-earth elements and fixing their positions in the periodic table.

**Derivation of Moseley’s law** — The general form of Moseley’s law can be easily derived from Bohr’s theory.

Let a transition occur from state $n_1$ to state $n_2$. So the energy of the emitted X-ray photon is

$$h\nu = RhcZ^2\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right) \text{ or } \nu = RcZ^2\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right) = a^2Z^2 \qquad \text{...(i)}$$

where $R$ is Rydberg constant, $a^2 = Rc\left(\frac{1}{n_1^2}-\frac{1}{n_2^2}\right)$ and $c$ the velocity of light in free space

$$\therefore \sqrt{\nu} = aZ$$

For $K_\alpha$-line, $n_1 = 1,\ n_2 = 2$, so that $a^2 = \frac{3}{4}Rc.$

The presence of the constant $b$ in Moseley equation, as deduced from experimental data, has been interpreted as the shielding effect of nucleus by the remaining electrons in the lower level and $b$ is known as the screening constant. For instance, for $K_\alpha$-emission one of the two K-electrons is first emitted leaving a single electron in K-shell. The negative charge of this residual electron partially screens the nuclear charge $+ Ze$ of the atom. The effective Coulomb force on the L-electron is thus due approximately to a net positive charge $+ (Z-1)e$. So instead of $Z^2$ in equation (i), we should rather write $(Z-1)^2$, that is to say,

$$\nu = \frac{3}{4}Rc(Z-1)^2 \text{ or } \sqrt{\nu} = a(Z-1)$$

where $a^2 = \frac{3}{4}Rc$, as before and $b = 1$.

With this interpretation, we have the Moseley’s law $\sqrt{\nu} = a(Z-b)$.

**Note 1.** For $K_\alpha$-line the screening constant $b = 1$. A plot of the frequency $\sqrt{\nu}/a$ against Z (Figure-11) shows from the intercept of the Z-axis that as a matter of fact $b \sim 1$.

**Note 2.** We know that $K_\alpha$, $K_\beta$-lines have doublet structure, a fact unknown at the time of Moseley’s discovery. So in Moseley’s diagram the mean frequencies of the doublet were plotted. Subsequent precise measurements showed that the Moseley diagram of $K_\alpha$-lines actually consists of two very close straight lines with slightly different slopes.

## 2.12. Absorption of X-rays

When a parallel beam of X-rays impinges on a slab of matter, most of it is transmitted through the material, but a part gets absorbed and the intensity of the beam emerging from the opposite side of the slab decreases. This absorption of X-rays leads to several phenomena. But they are collectively accounted for by what is called the absorption coefficients $\mu$ and $\mu/\rho$.

**Absorption coefficients**—Let $I$ be the intensity of a monochromatic X-ray beam that penetrates normally a thickness $dx$ of a material. Then $-dI/dx$ is the rate of change of intensity with thickness and negative sign indicate that intensity decrease with thickness.

But, experimentally, the rate of diminution of intensity is found to be proportional to the intensity $I$.

$$\therefore \quad -\frac{dI}{dx} \propto I$$

Assuming the material under discussion to be homogeneous, we get $\dfrac{dI}{dx} = -\mu I$ ...(i)

where $\mu$ is a characteristic constant of the material concerned. The constant $\mu$ is known as the linear absorption coefficient of the material.

From equation (i), we get $\dfrac{dI}{I} = -\mu dx$

Integrating, we get $\ell n\, I = -\mu x + A$

where $A$ is the constant of integration.

At $x = 0, I = I_0$, the original intensity of the incident beam. Therefore, $A = \ell n\, I_0$

$$\therefore \ell n\,(I/I_0) = -\mu x \Rightarrow I = I_0 e^{-\mu x} \qquad \text{...(ii)}$$

**Determination of $\mu$** —The linear absorption co-efficient $\mu$ of a material is usually measured by finding the thickness $x_h$ required to reduce the intensity to half its initial value.

From equation (ii), therefore, $\dfrac{I}{I_0} = e^{-\mu x_h} = \dfrac{1}{2}$

$$\therefore \mu = \frac{\ell n 2}{x_h} = \frac{0.693}{x_h}$$

Thus if $x_h$ is known, the co-efficients $\mu$ becomes known and $x_h$ is called half-value thickness.

Further, a plot of $\ell n\, I$ against $x$ (Figure-b) would give a straight line the slope of which is $-\mu$. From a measurement of this slope also, $\mu$ can be evaluated. The dimension of $\mu$ is plainly the inverse of length, $m^{-1}$.

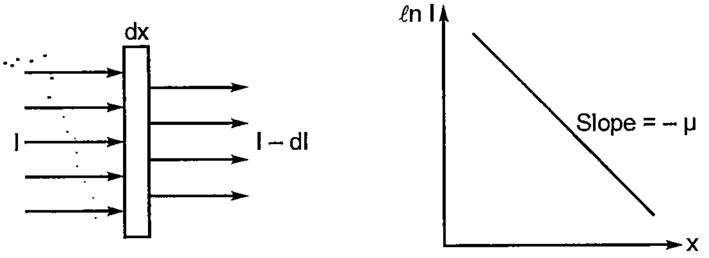


Fig. (a) Absorption of x-rays (b) ℓn I vs. x plot

It is by the value of $\mu$ that the quality of an X-ray beam is fixed. If $\mu$ is small and $x$ is large, the X-rays are said to be hard, if however $\mu$ is large, they are called soft.

**Mass absorption co-efficients** — It is found that $\mu$ varies rather widely for the same substance in their different physical states, e.g. ice, water and steam, due to the density factor. So $\mu$ is not a true constant for the substance. A true material constant, independent of its physical state, in this respect is the concept of mass absorption co-efficients as defined below.

Let $\rho$ be the density, $s$ the surface area, $x$ the thickness and $m$ the mass of the absorbing material. Then,

$$\rho = \frac{m}{sx}$$

$$\therefore \quad x = \frac{m}{\rho s}$$

Therefore, from (ii), we obtain : $I = I_0 e^{-\mu m/s\rho}$ ...(iii)

Plainly, $m/s$ is the mass per unit area of the absorber and the quantity $\mu/\rho$ $(= \mu_m)$ is called the mass absorption co-efficients of the absorber.

Therefore, Mass absorption co-efficients $\mu_m = \dfrac{\text{linear absorption coefficient}}{\text{density of the material}}$

The value of $\mu_m (= \mu/\rho)$ increases rapidly with atomic weight and thus explains why the Pb-screens shield x-radiations better than screens aluminium of other material of low atomic weights. X-rays or $\gamma$-ray radiography makes practical use of equation (iii), which accounts for the images of cavities in metal casting and so forth.

**Secondary radiations**—As already stated, the absorption of X-rays by matter leads to several phenomena. The part of the radiation removed from the incident beam (i.e. amount absorbed) produces, apart from heat, secondary radiations consisting mainly of the following.

1. **Photoelectrons** — The X-ray photons may be absorbed by electrons in inner shells (e.g. $K, L, M, ...$) which are thereby ejected photoelectrically with very high speed. The process is usually accompanied by the emission of characteristic fluorescent radiation due to electronic transition from outer shells. The net effect of the process is gradual diminution of the energy of the beam as it passes through the absorber.
2. **Scattered X-rays** — The incident X-rays may be scattered by the atomic electrons, and result in scattered X-rays of practically the same nature as the original radiation.
3. **Characteristic X-rays** — which depend only on the nature of the material but independent of the quality of the incident beam of X-radiation.

At higher energies, there could be photo-nuclear and other effects to attenuate the energy of X-radiation in its passage through the absorbers.

## 2.13. Variation of $\mu_m$ with wavelength : Absorption edges

The mass absorption coefficients, $\mu_m \propto Z^4$ where $Z$ is the atomic number of the material. So as $Z$ increases $\mu_m$ increases rapidly, as has already been stated.

$\mu_m$ also depends on the wavelength of X-radiation and the relation is $\mu_m \propto \lambda^3$. So as $\lambda$ increases, $\mu_m$ also increases rapidly. The variation of $\mu_m$ with $\lambda$ is represented graphically in (Figure-13), where in it is observed that as $\lambda$ increases, $\mu_m$ increases continuously up to a certain $\lambda$-value, followed by a sudden decrease in $\mu_m$ at a definite wavelength $\lambda_1 = \lambda_K$. The $\mu_m$ again rises continuously with $\lambda$ followed again by a sudden decrease at another definite wavelength $\lambda_2 = \lambda_{L_I}$, and immediately thereafter two more discontinuous – one at $\lambda_3 = \lambda_{L_{II}}$ and another at $\lambda_4 = \lambda_{L_{III}}$. The pattern gets more or less similarly repeated as $\lambda$ increases.

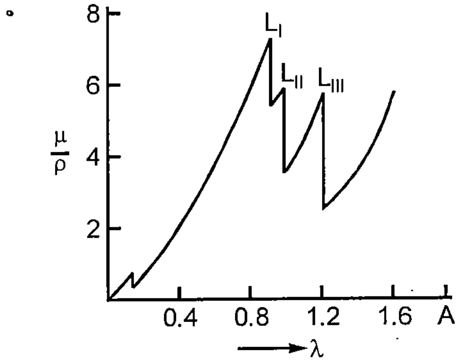


**Fig.13** Variation of $\mu_m$ with λ

The wavelength $\lambda_K$ corresponding to the first discontinuity in $\mu_m$-$\lambda$ curve is known as the *K*-absorption edge. The following discontinuities at $\lambda_{L_I}$, $\lambda_{L_{II}}$ and $\lambda_{L_{III}}$ are respectively called $\lambda_{L_I}$, $\lambda_{L_{II}}$ and $\lambda_{L_{III}}$ -absorption edges. The wavelengths of the different absorption edges are dependent on the absorber. Obviously, $\lambda_K < \lambda_{L_I} < \lambda_{L_{II}} < \lambda_{L_{III}}$. The wavelength $\lambda_K$ is slightly less than the maximum wavelength of the characteristic X-rays emitted by the absorber.

**Explanation for discontinuous**— The appearance of the discontinuous can be readily explained. So long the energy $E_v$ of the incident X-ray photons is greater than $E_K$, the binding energy of electron in the K-shell of absorbing the incident radiation with a kinetic energy.

$$\frac{1}{2}mv^2 = E_v - E_K$$

and the process may be termed photoelectric emission from the K-shell. As $E_v$ decreases, $\mu_m$ increases continuously till $E_v = E_K$. When $E_v$ becomes less than $E_K$, the incident radiation is not capable of knocking out K-electron any longer photoelectrically. So $\mu_m$ suddenly decreases giving rise to a discontinuity at $E_v = E_K$. However, even if $E_v < E_K$, the incident radiation can eject an L-electron as $E_L < E_K$. So with further increase of $\lambda$, $\mu_m$ rises again due to greater probability of absorption of the radiation by L-electrons. At $E_v = E_K$, this probability is maximum. Thereafter $\mu_m$ drops again since $E_v$ becomes less than $E_L$ and a second discontinuity is resulted. The pattern gets repeated in the same fashion.

**Absorption edges**—Unlike in the visible region, no discrete line appears in X-ray absorption spectrum. If an inner shell electron absorbs X-ray photon incident on it, it will be emitted as long as $E_v$ is greater than its binding energy. No transition of the ejected electron to an outer shell is possible as they are all filled up. Necessarily therefore incident radiation of all energies greater than the binding energy of the electron will be absorbed. This implies that x-radiations of all λ's less than a definite limiting value will be absorbed. The absorption spectrum thus shows a continuum with a sharp cur-off value on the long wavelength-end. It is this cur-off wavelength that corresponds to an absorption edge.

**Note 1.** Since the energies of different absorption edges (*K*, *L*, *M*,..., etc.) are equal to different binding energies ($E_K$, $E_L$, $E_M$... etc.) of the electrons in corresponding shells, the energy difference between different absorption edges will provide the energy differences of X-ray levels.

**Note 2.** Taking photoelectric emission and scattering into consideration, we may write $\mu = \mu_p + \mu_s$ where $\mu_p$ is the part contributed to $\mu$ due to photoelectric effect and $\mu_s$ that due to scattering. So the mass absorption coefficient, $\mu_m = \mu/\rho = (\mu_p + \mu_s)/\rho$.

The atomic absorption coefficient, $\mu_a$ is obtained by dividing the mass absorption coefficients by the number of atoms in unit mass of the absorber.

Therefore, atomic absorption coefficient, $\mu_a = \dfrac{\mu/\rho}{N_A/A}$ where $N_A$ is Avogadro number and $A$ the atomic weight of the absorber.

## 2.14. Scattering of X-rays : Thomson's theory

Let $E$ be the electric field of the incident unpolarised X-radiation. So the acceleration of the scatterer (electron) of charge $e$ and mass $m$ is

$$a = \frac{eE}{m} \qquad ...(i)$$

Under the influence of the incident radiation, the electron is set into forced vibration. According to the electromagnetic theory, it would emit electromagnetic waves of the same frequency, constituting the scattered radiation, in different directions from the scattering centre. If the velocity of the vibrating electron is small, compared to the light velocity, then it can be shown that the electric field of the scattered wave at a distance $r$ from the scattering centre in a direction $\beta$ with the incident electric vector $\vec{E}$ is

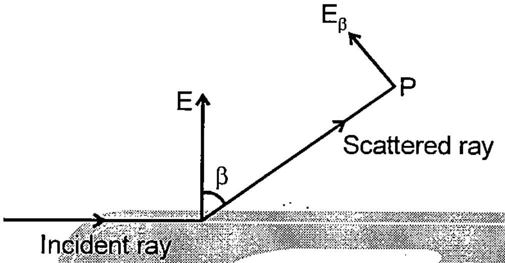


Fig.14-Scattering of x-rays

$$E'_\beta = \frac{ea}{4\pi\varepsilon_0 c^2 r}\sin\beta = \frac{e^2 E}{4\pi\varepsilon_0 c^2 m}\frac{\sin\beta}{r}, \text{ using equation (i)}$$

where $\varepsilon_0$ is the permitivity of free space.

By Poynting's theorem, the intensity of incident beam is given by $I_0 = \varepsilon_0 cE^2$ ...(ii)

and that of the scattered beam is $I'_\beta = \varepsilon_0 cE'^2_\beta = \dfrac{e^4 E^2}{16\pi^2 m^2 c^3 \varepsilon_0}\dfrac{\sin^2\beta}{r^2}$ ...(iii)

Therefore, $\dfrac{I'_\beta}{I_0} = \dfrac{e^4}{16\pi^2\varepsilon_0^2 m^2 c^4}\dfrac{\sin^2\beta}{r^2}$, using (ii) and (iii).

The incident radiation being unpolarised, we must take time-average of $\sin^2\beta$ in (iii). Let the incident radiation be along $x$-axis so that the electric vector $\vec{E}$ is in $yz$-plane (Figure-14a). Let $E_y$ and $E_z$ be the components of $\vec{E}$ along $y$-axis and $z$-axis respectively. Let the scattering centre be at $O$ and the direction of the scattered beam $OP$ be in $xy$-plane, making an angle $\theta$ with the incident beam. Then the electric field $E'_1$ and $E'_2$ of the scattered beam associated with components $E_y$ and $E_z$ of the incident electric vector $\vec{E}$ will be given by

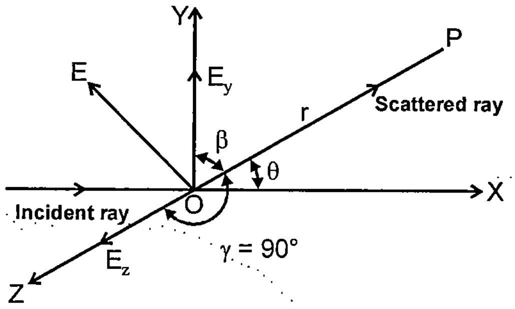


Fig.14(a)- Resolution of electric vector of incident beam

$$E'_1 = \frac{e^2 E_y \sin\left(\frac{\pi}{2}-\theta\right)}{4\pi\varepsilon_0 mc^2 r} = \frac{e^2 E_y}{4\pi\varepsilon_0 mc^2}\frac{\cos\theta}{r} \quad \left(\because\ \beta = \frac{\pi}{2}-\theta\right) \text{ and } E'_2 = \frac{e^2 E_z}{4\pi\varepsilon_0 mc^2 r}$$

Therefore, resultant electric intensity of the scattered wave $E'_\theta$ is $E'^2_\theta = E'^2_1 + E'^2_2$, since $E'_1$, $E'_2$ are mutually perpendicular.

Therefore, intensity of the scattered beam $I'_\theta$ is given by $I'_\theta = \varepsilon_0 c E'^2_\theta = \varepsilon_0 c \left( \frac{e^2}{4\pi\varepsilon_0 mc^2 r} \right)^2 (E_y^2 \cos^2\theta + E_z^2)$

whose time-average is $\overline{I}'_\theta = \frac{e^4}{16\pi^2 \varepsilon_0 m^2 c^3 r^2} (\vec{E}_y^2 \cos^2\theta + \vec{E}_z^2)$.

The time-averaged value of the incident intensity $\overline{I}_0 = \varepsilon_0 c \vec{E}^2 = \varepsilon_0 c \left( \vec{E}_y^2 + \vec{E}_z^2 \right)$.

Since the electric vector in the incident unpolarised beam is randomly oriented, $\vec{E}_y^2 = \vec{E}_z^2 = \frac{1}{2}\vec{E}^2 = \frac{1}{2\varepsilon_0 c}\overline{I}_0$.

Therefore, $I'_\theta = \frac{e^4}{32\pi^2 \varepsilon_0^2 m^2 c^4 r^2}(1 + \cos^2\theta)\, I_0$, omitting the bars.

This is the scattered intensity for a single electron. In a scatterer of thickness $\Delta x$ and unit cross-section, the number of scatterers (electrons) is $n\Delta x$ where $n$ is the number of electrons per unit volume of scatterer, each scattering independent of the other (incoherent scattering). So the resultant intensity of the scattered beam is

$$I_s = \frac{ne^4 \Delta x}{32\pi^2 \varepsilon_0^2 m^2 c^4 r^2}(1 + \cos^2\theta)\, I_0 \quad \text{...(iv)}$$

In deriving the above expression, the binding energy of electrons in the atom has been neglected (free electrons). Note from (iv) that the intensity is different in different directions, being minimum when $\theta = \pi/2$, that is, in a direction normal to incident radiation, when the scattered wave is plane polarized with electric vector along $z$.

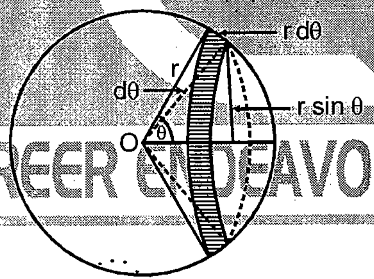


**Fig.15**-Computation of total scattered energy

To compute the total scattered energy, we are to integrate the above expression for $\theta$. Taking $r$ to be large compared to the dimension of the scatterer and assuming it to be concentrated at the origin $O$, let us draw (Figure-15) a sphere of radius $r$ about $O$. The area of the ring-shaped strip of the sphere between two cones of semi-vertical angles $\theta$ and $\theta + d\theta$ is

$$dS = (2\pi r \sin\theta)\, r d\theta$$
$$= 2\pi r^2 \sin\theta\, d\theta$$

∴ Total scattered energy,

$$W_s = \int I_s dS = I_0 \frac{ne^4 \Delta x}{32\pi^2 r^2 m^2 c^4 \varepsilon_0^2} 2\pi r^2 \int_0^x (1 + \cos^2\theta) \sin\theta\, d\theta = \frac{8\pi}{3} n \left( \frac{e^2}{4\pi\varepsilon_0 mc^2} \right)^2 I_0 \Delta x$$

∴ Scattering coefficient, $\sigma = \frac{W_s}{I_0 \Delta x} = \frac{8\pi}{3} n \left( \frac{e^2}{4\pi\varepsilon_0 mc^2} \right)^2$

This is the Thomson's scattering formula and this type of scattering is known as Thomson's scattering or unmodified scattering since there is no change in the frequency (or wavelength) of the rays.

The ratio $\sigma/n$ is called the classical scattering cross-section, $\sigma_T$, for a free electron.

$$\therefore \quad \sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{4\pi\varepsilon_0 mc^2}\right)^2 = \frac{8\pi}{3} r_0^2 \text{ where } r_0 = \frac{e^2}{4\pi\varepsilon_0 mc^2}$$ and is referred to as classical electron radius.

Since both $r_0$ and $\sigma_T$ depend on the three fundamental constants *e*, *m* and *c*, they are universal constants. It is easy to see that the dimension of $\sigma_T$ is of area and that of $r_0$ $(= 2.81 \times 10^{-15}\,\text{m})$ is of length.

**Note.** Thomson's theory of X-ray scattering has been verified by a number of workers, notably by Barkla, Hewlet, Coade, Compton, Hagenow and others.

**Application of Thomson's formula**—Thomson's formula was applied to compute the value of Z and the number of electrons per atom from an experimental evaluation of $\sigma$, by the British physicist Barkla.

We have : $n = \dfrac{3\sigma}{8\pi r_0^2} = \dfrac{3\sigma}{8\pi \times (2.81\times10^{-15})^2} = \dfrac{\sigma}{6.62\times10^{-29}}$

Barkla obtained the value of $\sigma/\rho$ for carbon equal to 0.02 m²/kg ($\sigma$ expressed in $\text{m}^{-1}$ and $\rho$ in kg/m³)

$$\therefore \quad n = \frac{0.02\rho}{6.62\times10^{-29}} = 3\times10^{26}\rho$$

The number of carbons atoms per unit volume is $n_a = \dfrac{N_A\rho}{12} = \dfrac{6.025\times10^{23}\times\rho\times10^3}{12} = 5.02\times10^{25}\rho.$

$\therefore$ Number of electrons per atom in carbon is given by $Z = \dfrac{n}{n_a} = \dfrac{3\times10^{26}\rho}{5.02\times10^{25}\rho} \simeq 6.$

This is the same as the atomic number of the element carbon in the periodic table. This implies that atomic number of carbon is equal to the number of electrons per atom. This was found to be valid for other elements as well. Later experiments of Chadwick of England with scattering of $\alpha$-particles vindicated that the number of positive charge in the atomic nucleus is also equal to the atomic number of the element. So, it follows that the nucleus charge of an atom is equal to the total negative charge of the atomic electrons.

## 2.15. Compton effect or Compton scattering

When a beam of monochromatic X-rays is scattered by a light element such as carbon, the scattered radiation in a particular direction consists not only of the unmodified incident radiation, as it excepted from the classical theory, but also of another new modified radiation of slightly shorter frequency. This type of X-ray scattering is called Compton scattering or Compton effect after the name of its discoverer A. H. Compton of the United States of America.

It was found that the wavelength shift, $d\lambda$, between the two radiations varies with the angle of scattering, but is independent of the wavelength of the primary beam and also the nature of the scatterer.

**Explanation**—On the basis of the wave theory, the phenomenon is not intelligible. The result however can be explained readily if, as suggested by Compton, taking the corpuscular view, the process is regarded as a collision between a photon and an electron with the laws of conservation of momentum and energy holding good. For, if a photon of energy $h\nu$ strikes an electron it will communicate its kinetic energy to the electron and thereby will itself lose energy. The scattered photon would then have a smaller energy $h\nu'$ and so smaller frequency $\nu'$.

**Calculation of wavelength-shift**—Let the energy $E$ of the photon before collision be $h\nu$. So its momentum $M$ before collision is $h\nu/c$ where $c$ is the velocity of light in vacuo. Let the energy $E'$ of the photon after collision be $h\nu'$. So its momentum $M'$ after collision is $h\nu'/c$.

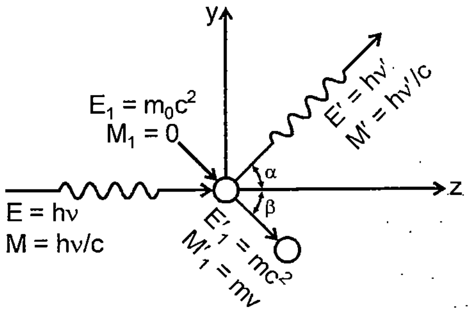


**Fig.16**-Compton effect

For simplicity, we regard the electron at rest previous to collision so that its energy $E_1$ is $m_0c^2$ corresponding to the rest mass $m_0$ and its momentum $M_1$ is zero. Let $v$ be the velocity of the electron after collision so that its momentum $M_1'$ is $mv$ and its energy $E_1' = mc^2$.

If α be the angle of scattering and β the angle of derivation of the electron, then from the laws of conservation of momentum and energy, we have

$$\left.\begin{aligned}\frac{h\nu}{c}&=\frac{h\nu'}{c}\cos\alpha+mv\cos\beta\\ 0&=\frac{h\nu'}{c}\sin\alpha-mv\sin\beta\end{aligned}\right\}\ \ldots\ \text{Momentum conservation}\qquad\ldots(\text{i})$$

$$\text{and}\quad h\nu+m_0c^2=h\nu'+mc^2\quad\ldots\ \text{Energy conservation}\qquad\ldots(\text{ii})$$

$$\text{From equation (i)}\quad\left.\begin{aligned}mvc\cos\beta&=h(\nu-\nu'\cos\alpha)\\ mvc\sin\beta&=h\nu'\sin\alpha\end{aligned}\right\}\qquad\ldots(\text{iii})$$

Squaring the equations in (iii) and adding $m^2v^2c^2=h^2(\nu^2+\nu'^2-2\nu\nu'\cos\alpha)$ ...(iv)

From equation (ii) again $mc^2=h(\nu-\nu')+m_0c^2$

$$\Rightarrow\quad m^2c^4=h^2(\nu^2+\nu'^2-2\nu\nu')+2m_0c^2h(\nu-\nu')+m_0^2c^4\qquad\ldots(\text{v})$$

Substracting (iv) from (v),

$$m^2c^4\left(1-\frac{v^2}{c^2}\right)=-2h^2\nu\nu'(1-\cos\alpha)+2m_0c^2h(\nu-\nu')+m_0^2c^4$$

$$\text{or,}\quad m_0^2c^4=-2h^2\nu\nu'(1-\cos\alpha)+2m_0c^2h(\nu-\nu')+m_0^2c^4\quad\left(\because\ m=m_0\big/\sqrt{1-v^2/c^2}\right)$$

$$\therefore\ 2h^2\nu\nu'(1-\cos\alpha)=2m_0c^2h(\nu-\nu')\ \text{ or }\ (1-\cos\alpha)=\frac{m_0c^2}{h}\frac{(\nu-\nu')}{\nu\nu'}=\frac{m_0c^2}{h}\left(\frac{1}{\nu'}-\frac{1}{\nu}\right)\qquad\ldots(\text{vi})$$

$$\text{But}\quad d\lambda=\lambda'-\lambda=c\left(\frac{1}{\nu'}-\frac{1}{\nu}\right)$$

$$\therefore\ d\lambda=\frac{h}{m_0c}(1-\cos\alpha)=\frac{h}{m_0c}2\sin^2\alpha/2,\quad\text{using (vi),}$$

$$=2\lambda_0\sin^2\alpha/2\qquad\ldots(\text{vii})$$

where $\lambda_0=h/m_0c$ is a universal constant as it involves the fundamental constants $h$, $c$ and $m_0$. It has the dimension of length and is known as the Compton wavelength of the electron having a value 0.024 Å.

**Dicussion**—The equation (vii) shows that the Compton shift $d\lambda$ is a function of $\alpha$, the angle of scattering and is independent of the wavelength $\lambda$ of the incident radiation and also the nature of the scatterer. The main features of the phenomenon are thus established.

If $\alpha = 90°$, the expression gives $d\lambda = h/m_0c = 0.024$ Å in agreement with the experimental value obtained by Compton.

When $\alpha = 0$, $d\lambda = 0$; and when $\alpha = 180°$, $d\lambda = 2\lambda_0 = 2 \times$ Compton wavelength and that is the maximum possible shift.

It therefore follows from (vi) that the Compton shift is most readily detected for radiation where $\lambda$ is not greater than few angstrom. For $\lambda = 5$Å, $d\lambda = 1\%$; for $\lambda = 1$Å, $d\lambda = 5\%$. Compton effect is also shown by $\gamma$-rays.

**Unmodified line**—The above treatment holds for a free electron, not bound to an atom. If the X-radiation collides with a bound electrons, the whole atom recoils as a result of Compton scattering and $m_0$ in the formula is to be replaced by $M_A$, the mass of the atom when it is seen from (vii) that $d\lambda \to 0$ for all values of $\alpha$ since $M_A \gg m_0$. This gives rise to the unmodified line observed by Compton. This conclusion has been verified by Ross using different elements as scatterer.

**Note.** We assumed in developing the relation (vii) that the electron is not only free but is also at rest. If, however, the electron be not at rest but in motion, the Compton radiation gets broadened.

- **Energy of scattered radiation and recoil electron**

We shall now calculate the energy of the scattered photon and the kinetic energy of the recoil electron.

**Energy of scattered photon**—From equation (vi) of Art. 2.15

$$\frac{1}{\nu'} - \frac{1}{\nu} = \frac{h}{m_0c^2}(1 - \cos\alpha)$$

$$\therefore \quad \nu' = \frac{1}{\dfrac{1}{\nu} + \dfrac{h}{m_0c^2}(1 - \cos\alpha)} \text{ or } h\nu' = \frac{h\nu}{\nu\left\{\dfrac{1}{\nu} + \dfrac{h}{m_0c^2}(1 - \cos\alpha)\right\}}$$

$$= \frac{h\nu}{1 + \dfrac{h\nu}{m_0c^2}(1 - \cos\alpha)} \qquad \text{...(i)}$$

This gives the energy of the scattered photon in terms of the angle of scattering $\alpha$ and the initial energy $h\nu$ of the incident photon.

**Kinetic energy of recoil electron**—Since the gain in electron energy is equal to the loss in photon energy, the kinetic energy of the recoil electron is given by

$$E_K = h\nu - h\nu'$$

$$= h\nu - \frac{h\nu}{1 + \dfrac{h\nu}{m_0c^2}(1 - \cos\alpha)}, \quad \text{using equation (i)}$$

$$= h\nu = \left\{1 - \frac{1}{1 + \dfrac{h\nu}{m_0c^2}(1 - \cos\alpha)}\right\} = h\nu = \frac{\dfrac{h\nu}{m_0c^2}(1 - \cos\alpha)}{1 + \dfrac{h\nu}{m_0c^2}(1 - \cos\alpha)}$$

which is the expression for the kinetic energy of recoil electron.

$$\therefore\ (E_K)_{\max} = \frac{h\nu}{\left(1+\dfrac{m_0c^2}{2h\nu}\right)}, \text{ when } \alpha = 180°.$$

**Relation between angle of scattering and angle of deviation** — From the set of equations (iii)

$$\tan\beta = \frac{\nu'\sin\alpha}{\nu-\nu'\cos\alpha} = \frac{\sin\alpha}{\nu/\nu'-\cos\alpha} \qquad \text{...(ii)}$$

$$\text{Again, } \frac{1}{\nu'} = \frac{1}{\nu} + \frac{h}{m_0c^2}(1-\cos\alpha)$$

$$= \frac{1}{\nu}\left\{1+\frac{h\nu}{m_0c^2}(1-\cos\alpha)\right\} = \frac{1}{\nu}(1+2a\sin^2\alpha/2),$$

$$\text{where } a = \frac{h\nu}{m_0c^2} \quad \therefore\ \nu' = \frac{\nu}{1+2a\sin^2\alpha/2}$$

Substituting this value of $\nu'$ in (ii) above,

$$\tan\beta = \frac{2\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}}{1+2a\sin^2\dfrac{\alpha}{2}-\cos\alpha} = \frac{2\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}}{2a\sin^2\dfrac{\alpha}{2}+2\sin^2\dfrac{\alpha}{2}}$$

$$\therefore\ \cot\beta = (1+a)\tan\frac{\alpha}{2} = \left(1+\frac{h\nu}{m_0c^2}\right)\tan\frac{\alpha}{2}$$

which is the relation between the angle of scattering, $\alpha$ of the photon and the angle of deviation $\beta$ of the recoil electron.

## SOLVED PROBLEMS

1. An X-ray tube operates at 30 kV. Calculate the shortest wavelength ofthe emitted X-rays.

**Soln.** Energy of electrons generating the X-rays = Ve. When the above energy is completely converted to X-rays, maximum frequency $v_m$ or minimum wavelength $\lambda_0$ is obtained.

$$\therefore \quad Ve = hv_m = \frac{hc}{\lambda_0} \qquad \therefore \lambda_0 = \frac{hc}{Ve}$$

Now, $V = 30\times10^3\ volt, h = 6.62\times10^{-34} Js, e = 1.602\times10^{-19} C$

$$\therefore \quad \lambda_0 = \frac{6.62\times10^{-34}\times3\times10^8}{1.602\times10^{-19}\times30\times10^3} = 4.132\times10^{-11} m = 0.4132 A$$

2. Electrons bombarding the anode of Coolidge tube produce X-rays of wavelength 1A. Find the energy of each electron at the moment of impact.

**Soln.** Let $v$ be the electron velocity at the time of impact with the anode.

Therefore, energy of electron $= \frac{1}{2}mv^2 = Ve = hv = \frac{hc}{\lambda}$

where, V is the freqency and $\lambda$ the wavelength of X-rays, V the operation voltage of the tube, e the eelctronic charge.

Therefore, energy $= \frac{hc}{\lambda} = \frac{6.62\times10^{-34}\times3\times10^8}{1\times10^{-10}} = 19.86\times10^{-16} J = 1.24\times10^4\ eV$

3. The intensity of a beam of X-rays is reduced to 90% of its initial value in passing through a slab of beryllium 3 mm thick and of density $1.84\times10^3\ kg/m^3$. Calculate the mass absorption coefficient.

**Soln.** Here, $\frac{I}{I_0} = 90\% = 0.9$; x = 3 mm $= 3\times10^{-3}$ m and $\rho = 1.84\times10^3\ kg/m^3$

From, $\frac{I}{I_0} = e^{-\mu x}$, we get

$$0.9 = e^{-3\times10^{-3}\mu}$$

$$\therefore \quad 3\times10^{-3}\mu = -\ln 0.9 = 0.105 \qquad \therefore \mu = 35$$

Therefore, mass absorption coefficient, $\mu_m = \frac{\mu}{\rho} = \frac{35}{1.84\times10^3} = 19\times10^{-3} m^2kg^{-1}$

4. An X-ray photon of wavelength 0.1 A is reflected at an angle 90° with its original direction after collision with an electron at rest. Find the energy it loses on collision.

**Soln.** Incident wavelength, $d\lambda = \frac{h}{m_0c} = 0.024$ Å

Therefore, wavelength of scattered radiation, $\lambda' = 0.1 + 0.024 = 0.124 Å = 0.124\times10^{-10} m$

Incident frequency, $v = \frac{c}{\lambda} = 3\times10^8 / 0.1\times10^{-10} = 3\times10^{19} Hz$

Scattered frequency, $v' = \frac{c}{\lambda'} = 3\times10^8 / 0.124\times10^{-10} = 2.419\times10^{19} Hz$

Therefore, energy loss in collision is $\Delta E = h\nu - h\nu' = h(\nu - \nu') = 6.6\times10^{-34}(3-2.419)\times10^{19}$

$$= 6.6\times0.581\times10^{-15}J = 2.39\times10^{4}eV$$

5. Find the critical voltage that must be applied to an X-ray tube to excite the K-series of copper, given that the K-absorption limit is 1.380Å

**Soln.** Here, $\lambda_{\min} = 1.380\times10^{-10}m$. From the relation, $Ve = \frac{hc}{\lambda}$

$$Ve = \frac{hc}{e\lambda} = \frac{6.62\times10^{-34}\times3\times10^{8}}{1.602\times10^{-19}\times1.38\times10^{-10}} = 8.98\,kV$$

6. Calculate the fractional change in the wavelength of an X-ray of wavelength 0.400 Å that undergoes a 90° Compton scattering from an electron.

**Soln.** $\lambda' - \lambda = \frac{h}{M_0c}(1-\cos\theta) = (0.0243\text{ Å})(1-\cos 90^\circ) = 0.0243\text{Å}$

$$\frac{\lambda'-\lambda}{\lambda} = \frac{0.0243\text{ Å}}{0.400\text{ Å}} = 0.0608$$

7. An X-ray of wavelength 0.300 Å undergoes a 60° Compton scattering. Find the wavelength of the scattered photon and the energy of the electron after the scattering.

**Soln.** $\lambda' = \lambda + \frac{h}{m_0c}(1-\cos\theta) = 0.30\text{ Å} + (0.0243\text{ Å})(1-\cos 60^\circ) = 0.312\text{ Å}$

From energy conservation,

$$\frac{hc}{\lambda} + m_0c^2 = \frac{hc}{\lambda} + K_e + m_0c^2 \quad or \quad \frac{12.4\text{ keV Å}}{0.3\text{ Å}} = \frac{12.4\text{ keV Å}}{0.312\text{ Å}} + K_e \Rightarrow K_e = 1.59\text{ keV}$$

8. In Compton scattering the angle of scattering $\phi$ of the photon and the change in wave length $\Delta\lambda$ are related by $\Delta\lambda = \frac{h}{m_0c}(1-\cos\phi)$

If the incident photon have wavelength $2.0\times10^{-10}$m and $\phi = 90^\circ$, deduce

(i) the wavelength of the scattered photon

(ii) the energy of the recoil electron and

(iii) the angle at which the recoil electron appears.

**Soln.** If $\lambda'$ and $\lambda$ are the wavelengths of scattered and incident photons, we have

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_0c}(1-\cos\phi) = \frac{6.6\times10^{-34}}{9.1\times10^{-31}\times3\times10^{8}}(1-\cos 90^\circ)$$

$$= \frac{6.6\times10^{-34}}{9.1\times10^{-31}\times3\times10^{8}}(1-0) = 2.4\times10^{-12}\text{m.} = 0.024\text{ Å}$$

(ii) Neglecting the binding energy of the recoil electron the energy of the recoil electron is given by

$$T=(m-m_0)c^2=h\nu-h\nu'=h(\nu-\nu')=hc\left(\frac{1}{\lambda}-\frac{1}{\lambda'}\right)$$

$$=hc\frac{(\lambda'-\lambda)}{\lambda\lambda'}=6.6\times10^{-34}\times3\times10^{8}\left[\frac{2.024\times10^{-10}-2\times10^{-10}}{2.024\times10^{-10}\times2\times10^{-10}}\right]\text{J}$$

$$=\frac{6.6\times10^{-34}\times3\times10^{8}\times0.024\times10^{-10}}{2.024\times10^{-10}\times2\times10^{-10}}\text{ J}$$

$$=1.17\times10^{-17}\text{ J}$$

(iii) The angle $\theta$ at which the recoil electron appears is given by

$$\tan\theta=\frac{\lambda\sin\phi}{\lambda'-\lambda\cos\phi}=\frac{2.0\times10^{-10}\sin90^\circ}{2.024\times10^{-10}-2.0\times10^{-10}\cos90}=\frac{2.0\times10^{-10}}{2.204\times10^{-10}}=\frac{2}{2.024}$$

$$\tan\theta=0.9980;\quad \theta=\tan^{-1}(0.9980);\quad \theta=44^\circ57'$$

9. A beam of gamma radiation having photon energy of 510 keV is incident on a foil of aluminium. Calculate the wavelength of the radiation incident at 90° and also the energy and direction of the emission of the corresponding electron.

**Soln.** Energy of each photon $=h\nu=\dfrac{hc}{\lambda}=510\text{ keV}$

i.e. $\dfrac{hc}{\lambda}=510\times10^{3}\times1.6\times10^{-19}\text{ joule};\qquad \Rightarrow \lambda=\dfrac{hc}{510\times10^{3}\times1.6\times10^{-19}}$

$$=\frac{6.6\times10^{-34}\times3\times10^{8}}{510\times5.6\times10^{-16}}=2.426\times10^{-12}\text{ m}$$

According to Compton scattering

$$\lambda'-\lambda=\frac{h}{m_0c}(1-\cos\phi)\qquad\Rightarrow\lambda'=\lambda+\frac{h}{m_0c}(1-\cos\phi)$$

$$=2.426\times10^{-12}+\frac{6.6\times10^{-34}}{9.1\times10^{-31}\times3\times10^{8}}(1-\cos90^\circ)$$

$$=2.426\times10^{-12}+2.417\times10^{-12}=4.843\times10^{-12}\text{ m}$$

The energy of recoil electron is given by

$$E=h\nu-h\nu'=h(\nu-\nu')=hc\left(\frac{1}{\lambda}-\frac{1}{\lambda'}\right)=\frac{6.6\times10^{-34}\times3\times10^{8}\times2417\times10^{-12}}{2.426\times4.843\times10^{-24}}=4.072\times10^{-14}\text{ joules.}$$

The direction of recoil electron $\theta$ is given by

$$\tan\theta=\frac{\lambda\sin\phi}{\lambda'-\lambda\cos\phi}=\frac{2.426\times10^{-12}\sin90^\circ}{4.843\times10^{-12}-2.426\cos90^\circ}=\frac{2.426\times10^{-12}}{4.843\times10^{-12}}=\frac{2.426}{4.843}$$

$$\tan\theta=0.5008;\qquad \theta=\tan^{-1}(0.5008);\qquad \theta=26^\circ36'$$

0. In Compton scattering, wavelength shift is towards **[BHU 2010]**

(a) shorter wavelength
(b) longer wavelength
(c) shorter or longer depending on the angle of scattering
(d) shorter or longer side depending on photon energy

**Soln.** The incident photon loses its energy to the target particle and hence its wavelength increases.

**Correct option is (b)**

1. X-rays of 10.0 p.m. are scattered from a target. The maximum energy of the recoil electron (h/mc for electron is $2.426\times10^{-12}$ m) is **[BHU 2011]**

(a) $3.27\times10^{-15}$ joules
(b) $6.54\times10^{-15}$ joules
(c) $5.64\times10^{-15}$ joules
(d) $2.37\times10^{-15}$ joules

**Soln.** Energy of the incident x-ray photon $h\nu=\dfrac{hc}{\lambda}=\dfrac{1240\ eV-nm}{10\times10^{-3}\ nm}=124\ keV$

Maximum energy of the recoil electron will be

$$E_{k,\max}=\frac{h\nu}{1+\dfrac{m_0c^2}{2h\nu}}=\frac{124}{1+\dfrac{511}{248}}keV=40.52\ keV=6.48\times10^{-15}J$$

12. The atomic number Z of the element which has a $K_\alpha$ X-ray line of wavelength 0.180 nm (given that Rydberg constant $R=1.097\times10^7\ m^{-1}$) is **[B.H.U-2011]**

(a) 27
(b) 26
(c) 52
(d) 54

**Soln.** Using Moseley's law,

$$\bar{v}=\frac{3}{4}R(Z-\sigma)^2$$

where $\bar{v}$ is the wave number and $\sigma$ is screening constant. For $K_\alpha$ X-ray line $\sigma=1$.

$$\Rightarrow\ \frac{1}{\lambda}=\frac{3}{4}\times1.097\times10^7\times(Z-1)^2$$

$$\Rightarrow\ \frac{1}{0.180\times10^{-9}}=\frac{3}{4}\times1.097\times10^7(Z-1)^2$$

$$\Rightarrow\ (Z-1)^2=\frac{4\times10^9}{0.180\times3\times1.097\times10^7}$$

$$\Rightarrow\ (Z-1)^2=675.24$$

$$\Rightarrow\ (Z-1)=25.98\approx26\ \Rightarrow\ Z=27$$

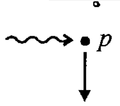
**Correct option is (a)**

13. Photons of wavelength $\lambda$ scatter elastically from free protons initially at rest. The wavelength of the protons scattered at 90° is increased by **[BHU 2012]**

(a) $\dfrac{\lambda}{137}$
(b) $\dfrac{h}{m_eC}$
(c) $\dfrac{h}{m_pC}$
(d) 0

where $m_e$ and $m_p$ are the rest mass of the electron and proton.

**Soln.** The shift in the wavelength is $\Delta\lambda = \dfrac{h}{m_p c}(1-\cos\theta)$, when $\theta = 90^\circ$, $\Delta\lambda = \dfrac{h}{m_p c}$.



**Correct option is (c)**

14. According to Moseley's law, the relation between the atomic number Z and frequency $\nu$ is given by

(a) $\nu \propto (Z-b)$ (b) $\nu \propto Z$ (c) $\nu \propto (Z-b)^3$ (d) $\nu \propto (Z-b)^2$ **[B.H.U-2013]**

**Soln.** According to Moseley's law,

$$\nu = \frac{3}{4}Rc(Z-\sigma)^2$$

where $\nu$ is the frequency, R is the Rydberg constant, c is the speed of light, $\sigma$ is the screening constant and Z is the atom number.

$\therefore \ \nu \propto (Z-b)^2$

**Correct option is (d)**

15. The Compton shift in X-ray wavelength depends on :

(a) Scattering angle only (b) Scattering angle and wavelength of X-ray both

(c) Wavelength of X-ray only (d) None of the (a), (b) and (c) **[B.H.U-2014]**

**Soln.** The Compton shift in X-ray wavelength is given by

$$\Delta\lambda = \frac{2h}{m_0 c}\sin^2\frac{\alpha}{2} \qquad \text{... (i)}$$

where $h$ is Planck's constant, $c$ is speed of light, $\alpha$ is the scattering angle, $m_0$ is the electron rest mass, From (i) one can conclude that the Compton shift in X-ray wavelength depends only on the scattering angle.

**Correct option is (a)**

6. X-ray spectrum of a cobalt larget ($Z=27$) contains strong $k_\alpha$ line of wavelength 0.1785 nm and a weak line having wavelength of 0.1537 nm due to impurities. The atomic number of impurity element is

(a) 29 (b) 24 (c) 23 (d) 30 **[B.H.U-2015]**

**Soln.** According to Moseley's law :

$$\bar{\nu} \propto (Z-\sigma)^2$$

where $\bar{\nu}$ is the wave number and $\sigma$ is screening constant

$$\frac{1}{\lambda} \propto (Z-\sigma)^2$$

For strong $K_\alpha$ line

$$\frac{1}{0.1785\ nm} \propto (27-1)^2 \qquad \text{... (i)}$$

For weak $K_\alpha$ line

$$\frac{1}{0.1537\ nm} \propto (Z-1)^2 \qquad \text{... (ii)}$$

For $K_\alpha$ lines $\sigma = 1$

Dividing (i) by (ii),

$$\frac{1}{0.1785\ nm} \div \frac{1}{0.1537\ nm} = \frac{26^2}{(Z-1)^2}$$

$$\Rightarrow \ (Z-1)^2 = \frac{26^2 \times 0.1785}{0.1537} = 785.075 \Rightarrow \ Z-1 \approx 28 \ \Rightarrow \ Z = 29$$

**Correct option is (a)**

7. Due to Compton scattering, the change in wavelength is **[BHU 2015]**
(a) dependent on electron charge
(b) dependent on electron mass and charge
(c) dependent on electron mass and scattering angle both
(d) dependent on scattering angle only

**oln.** The change in wavelength in Compton scattering is

$$\Delta\lambda = \frac{h}{m_0 c}(1-\cos\theta)$$

where $m_0$ is the rest mass of the electron and $\theta$ is the scattering angle.

**Correct option is (c)**

8. The change in the wavelength of the scattered light in the Compton scattering is equal to the Compton wavelength of the electron when: **[BHU 2016]**

(a) the deflection angle of photon is $\pi/2$ (b) the deflection angle of photon is 0
(c) the deflection angle of photon is $\pi/3$ (d) the deflection angle of photon is $\pi$

**oln.** The change in the wavelength of the scattered light is

$$\Delta\lambda = \frac{h}{m_0 c}(1-\cos\theta)$$ , where $\theta$ is the deflection angle.

Since, wavelength shift is equal to compton wavelength, then

$$\Delta\lambda = \frac{h}{m_0 c} \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

**Correct option is (a)**

9. The maximum value of the change in the wavelength of the scattered light in the case of Compton scattering is:
(a) 0.024 Å (b) 0.048 Å (c) 0.0 Å (d) 0.072 Å **[BHU 2016]**

**oln.** The change in the wavelength of the scattered light is $\Delta\lambda = \frac{h}{m_0 c}(1-\cos\theta)$.

The change would be maximum when $\cos\theta = -1$.

Therefore, $(\Delta\lambda)_{\max} = \frac{2h}{m_0 c} = 0.048$ Å

**Correct option is (b)**

0. In a Compton scattering experiment, the recoil electron has to be :
(a) relativistic (b) non-relativistic **[BHU 2016]**
(c) Sometimes relativistic (d) Sometimes non-relativistic

**oln.** In Compton scattering experiment, the incident photon has energy of order of keV and hence the recoil electron also has the kinetic energy of order of keV. Moreover, the rest mass energy of the electron is of order of keV. So the rest mess energy of the electron is comparable to its kinetic energy. This is why the recoil electron is relativistic in nature.

**Correct option is (a)**

21. If the incident photon has a wavelength equal is 2Å and its angle of deflection is 90°, the kinetic energy of the recoil electron in the Compton scattering would be **[BHU 2016]**

(a) $1.17\times10^{-17}$ ergs
(b) $1.17\times10^{-17}$ joules
(c) $1.17\times10^{-17}$ KeV
(d) $1.17\times10^{-17}$ eV

**Soln.** Energy of the incident photon $h\nu=\frac{hc}{\lambda}=\frac{1240\,eV-nm}{0.2\,nm}=6.2\,keV$

Kinetic energy of the recoil electron

$$E_k=\frac{\frac{h^2\nu^2}{m_0c^2}(1-\cos\alpha)}{1+\frac{h\nu}{m_0c^2}(1-\cos\alpha)}=\frac{\frac{(6.2)^2}{(511)}(1-0)}{1+\frac{6.2}{511}(1-0)}keV=74.11\,eV=1.18\times10^{-17}\,J$$

**Correct option is (b)**

22. In the Compton scattering there is an **[BHU 2016]**
(a) inelastic collision of a photon with a free $e^-$
(b) inelastic collision of a photon with a proton
(c) inelastic collision of a photon with a neutron
(d) inelastic collision of a photon with a bound electron of an atom

**Soln.** A photon collide inelastically with a free electron in the Compton scattering.

**Correct option is (a)**

23. In the case of Rutherford scattering, if the impact parameter is zero, the deflection of the α-particle would be by the : **[B.H.U-2016]**

(a) Angle $\frac{\pi}{2}$ (b) Angle $\frac{\pi}{4}$ (c) Angle $\pi$ (d) Angle $2\pi$

**Soln.** In case of Rutherford scattering, impact parameter $b$ is given by, $b=\frac{Zke^2}{2(K.E)}\cot\frac{\theta}{2}$

where $Z$ is the atomic number of target nuclei, $k=\frac{1}{4\pi\in_0}$, $e$ is the electron charge, K.E, is the kinetic energy of α-particle and θ is the scattering angle.

Now, $b=0 \Rightarrow \cot\frac{\theta}{2}=0 \Rightarrow \frac{\theta}{2}=\frac{\pi}{2} \Rightarrow \theta=\pi$

**Correct option is (c)**

24. Rutherford scattering is important because it sheds light on **[B.H.U-2016]**
(a) the structure of the molecule
(b) the size of the molecule
(c) the quantum nature of the molecule
(d) the structure of the atom

**Soln.** Conclusions drawn from Rutherford scattering experiment:
(i) an atom was much more than just empty space and scattered electron.
(ii) an atom must have a positively charged center that contains most of its mass.
Therefore, Rutherford scattering is important because it sheds light on the structure of the atom.
**Correct option is (d)**

25. Separate Compton effect experiments are carried out using visible light and X-rays. If the scattered radiation is observed at the same scattering angle, then **[HCU 2012]**

(a) the X-rays have greater shift in wavelength and greater change in photon energy.

(b) the two radiations have the same shift in wavelength and X-rays have a greater change in photon energy.

(c) the two radiations have the same shift in wavelength and visible light has greater change in photon energy

(d) the two radiations have the same shift in wavelength and the same change in photon energy.

**Soln.** The change in the wavelength of the incident radiation is

$$\Delta\lambda = \frac{h}{m_0 c}(1-\cos\theta)$$

Since the change depends only on the scattering angle, the change in the wavelength of both the radiation would be same.

$$\Rightarrow \lambda_f - \lambda_i = \frac{h}{m_0 c}(1-\cos\theta)$$

$$\Rightarrow \lambda_f = \lambda_i - \frac{h}{m_0 c}(1-\cos\theta) \qquad \text{...(i)}$$

The change in the photon energy is $\Delta E = hc\left(\frac{1}{\lambda_i} - \frac{1}{\lambda_f}\right) = \frac{hc}{\lambda_i}\frac{\Delta\lambda}{\lambda_f} = \frac{hc\,\Delta\lambda}{\lambda_i(\lambda_i+\Delta\lambda)}$

Since, $(\lambda_i)_{\text{visible}} > (\lambda_i)_{\text{X-ray}}$

$(\Delta E)_{\text{X-ray}} > (\Delta E)_{\text{visible}}$

**Correct option is (b)**

26. A scientist performing the Compton scattering experiment finds that the incident wavelength $\lambda_1$ is shifted by 1.5 percent for the scattering angle of 120°. The waelength $\lambda_1$ is given by **[HCU 2013]**

(a) 0.242 nm (b) 2.420 nm (c) 24.200 nm (d) 0.024 nm

**Soln.** The according to compton scattering, the wavelength of the scattering light is given by

$$\lambda' = \lambda + \frac{h}{m_e c}(1-\cos\theta) \Rightarrow \lambda' = \lambda_1 + \frac{h}{m_e c}(1-\cos 120^\circ)$$

$$\Rightarrow \lambda' = \lambda_1 + \frac{h}{m_e c}1.5 \qquad \Rightarrow \lambda' - \lambda_1 = \frac{h}{m_e c}1.5$$

$$\text{Given}: \Delta\lambda = \lambda' - \lambda_1 = \frac{15}{100}\times\lambda_1 = \frac{h}{m_e c}\times 1.5$$

$$\Rightarrow \lambda_1 = \frac{h}{m_e c}\times 10 = \frac{6.62\times 10^{-3}}{9.1\times 10^{-31}\times 3\times 10^{8}} = 0.24\times 10^{-10}\text{ m} = 0.024\text{ nm.}$$

**Correct option is (d)**

## PRACTICE SET

1. What is the wavelength of X-ray emitted when 100 kV electrons strike a target? What is the frequency?

[0.124Å; $24.2\times10^{18}$]

2. The mass absorption coefficient of silver is 3.8 $m^2kg^{-1}$ for X-rays of $\lambda = 0.4\,\text{Å}$. find the atomic mass absorption coefficient of silver. [$6.087\times10^{-25}$ $m^2$/atom]

3. Determine the maximum speed attained by electrons in an X-ray tube operating at 10 kV.

[$5.964\times10^7$ m/s]

4. A photon with energy 0.46 MeV is scattered at an angle 120° by a free electron at rest. Find (i) the energy of the scattered photon and (ii) the energy transferred to the electron.

[0.20 MeV; 0.26 MeV]

5. When photons of wavelength 0.024Å are incident on a target, scattered photons are detected at an angle 60°. Calculate the wavelength of the scattered photon and the angle at which it is scattered.

[0.036 Å; 40.9°]

# Chapter 3

# Wave Properties of Particle

## 3.1 Wave particle Duality:

According to classical mechanics, the particle can behave as a particle and wave can behave like a wave. Particle is characterized by energy and momentum and wave is characterized byamplitude and wave vector. In 1923, De-broglie postulated that a moving particle can behave as a wave. The de-broglie wavelength associated with a particle of rest mass '$m_0$' moving with a velocity 'v' is given by

$$\lambda = \frac{h}{p} = \frac{h}{\gamma m_0 v}, \qquad \text{where } \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \text{ is Lorentz factor}$$

This hypothesis is experimentally verified by Davission-Germer experiment in 1927. If the De-broglie wavelength is comparable to size of the particle, then the wave nature of the particle can be experimentally detected. Otherwise, if the De-broglie wavelength is very small, the wave nature cannot be detected experimentally.

Schrodinger modified the concept wave nature of matter i.e. particle is not like a single plane wave but is like a group of waves i.e Wave packet. It is just the superposition of waves different wavelengths and frequencies.

**Phase Velocity:**

A plane wave travelling along positive direction of x-axis can be represented as,

$$\psi(x,t) = A\sin(\omega t - kx) = A\sin\omega\left(t - \frac{x}{v}\right)$$

where, $\omega$ is the angular frequency, $v$ is the velocity and $A$ is the amplitude of the wave.

The phase $\phi$ of the wave at any time 't' at a distance 'x' from the origin is $\phi(x,t) = (\omega t - kx)$. Phase velocity $v_p$ of the wave is the velocity with which displacement of a given phase, moves in the forward direction. It is also the velocity of a monochromatic wave with which crest or trough of the wave travels in a medium. In a dispersive medium, $v_p$ is different for different wavelengths. The phase velocity of the propagating monochromatic wave is given by, $v_p = \frac{\omega}{k} = \frac{E}{p}$.

From the relativistic mass energy relation, $v_p = \frac{E}{p} = \frac{\gamma m_0 c^2}{\gamma m_0 v} = \frac{c^2}{v}$

Since, the phase velocity of a de-broglie wave is always greater than the velocity of light in free space. But according to the special theory of relaticvity, no energy can be transmitted with velocity greater than c.

This suggests that the phase velocity has no physical significance and we can not respresent the wave nature of a moving particle by a single plane wave.

**Group velocity:**

A wave group or wave packet is described as the superposition of individual waves of different wavelengths. These waves interfere with each other and results in variation of amplitude. These defines the wave group shape. If the medium is non-dispersive, then phase velocity of the individual waves will not change with wavelength. As a result, wave packet travels with common phase velocity. But, in a dispersive medium, phase velocity of individual waves will be different. As a result, wave packet moves with a different velocity than phase velocity which is known as group velocity '$v_g$'.

Let, the wave group is formed by the superposition of two monochromatic waves having same amplitude but differing in angular frequency by an amount '$\Delta\omega$' and in wave number '$\Delta k$' i.e

$$\psi_1 = A\sin(\omega t - kx)$$

$$\psi_2 = A\sin[(\omega + \Delta\omega)t - (k + \Delta k)x]$$

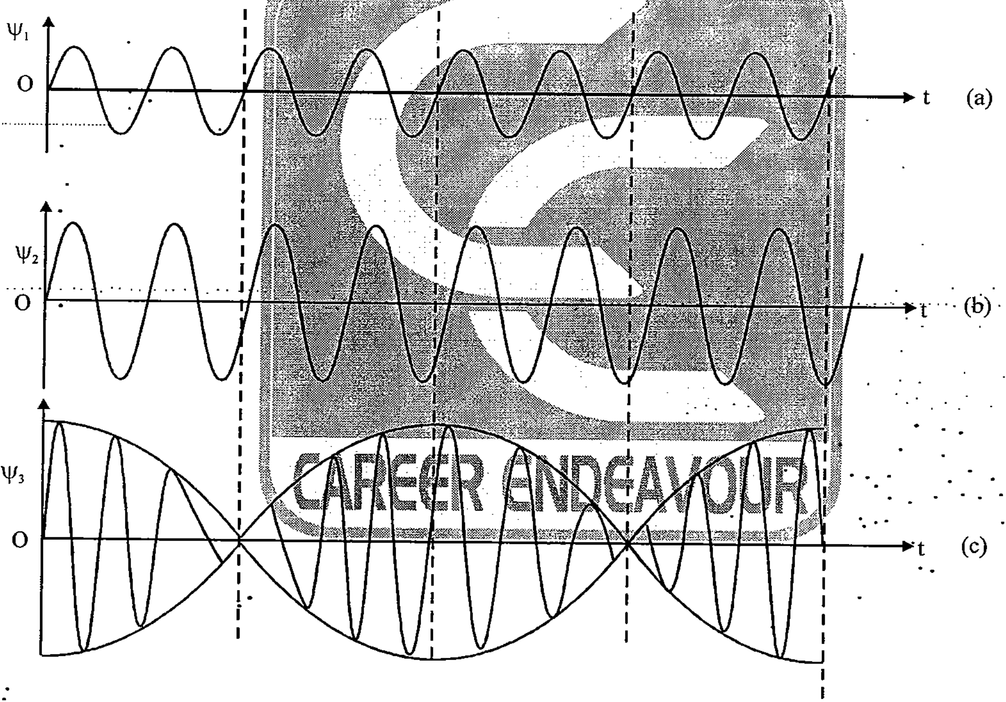


**Figure :** Superposition of two sinusoidal waves of slightly different frequncies [(a) and (b)] Curve (c) represents the resultant motion.

The resultant wave i.e. wave packet can be written as,

$$\psi = 2A\cos\left[\frac{1}{2}(\Delta\omega t - \Delta kx)\right]\sin(\omega t - kx)$$

Now, group velocity is the velocity with which maximum amplitude or the envelope of the wave packet moves and it is given as: $v_g = \frac{d\omega}{dk} = \frac{dE}{dp}$.

From the relativistic mass-energy relation :

$$E^2 = p^2c^2 + m_0^2c^4 \qquad \Rightarrow 2E\frac{dE}{dp} = 2pc^2$$

Therefore, $$v_g = \frac{dE}{dp} = \frac{pc^2}{E} = \frac{\gamma m_0 v c^2}{\gamma m_0 c^2} = v$$

**Relation between Phase velocity and Group velocity:** $v_g = v_p - \lambda\dfrac{dv_p}{d\lambda}$

For, non-dispersive medium, all waves travel with same phase velocity: $\dfrac{dv_p}{d\lambda} = 0 \Rightarrow v_g = v_p$

Dispersive medium are of two types:

(i) Normal dispersive medium: $\dfrac{dv_p}{d\lambda} = +ve \Rightarrow v_g < v_p$

(ii) Anamolous dispersive medium: $\dfrac{dv_p}{d\lambda} = -ve \Rightarrow v_g > v_p$

### 3.2 Heisenberg's Uncertainty principle:

According to classical mechanics, a moving particle has a definite momentum and position in space. Both the position and momentum of such a particle can be definitely determined. Recently, the classical view has been proved to be an approximation and is inadquate for describing the dynamical behaviour of the macroscopic objects. It cannot describe satisfactorily the behaviour of sub-atomic particles.

In quantum mechanics, a particle is described by a wave packet of linear extension $(\Delta x)$ which surrounds the position of the classical particle and moves with group velocity. According to Max Born's probability interpretation, the particle may be found everywhere within the wave packet. This means that the position of the particle is uncertain within the limits $(\Delta x)$ of the wave packet. So, we may say that the particle may be found anywhere within $(\Delta x)$. Again; a wave packet is formed by the superposition of large number of waves having different wavelengths, the wavelength the wave packet cannot be said definitely. What we can say that the wavelength of the wave packet lies within certain range $\lambda$ to $\lambda + \Delta\lambda$. As the momentum of a particle is related to wavelength by

$$p = \frac{h}{\lambda}, \Rightarrow \Delta p = -\frac{h}{\lambda^2}\Delta\lambda$$

the particle momentum is also within p to $p + \Delta p$. So, we cannot definitely say what is the momentum of the particle. Hence, because of the wave nature of partice there is uncertainty $(\Delta x)$ in position of the particle as well as uncertainty in corresponding momentum $(\Delta p)$.

If the number of superposed wave as well as their range of wavelength $(\Delta\lambda)$ be increased, then the linear speed $(\Delta x)$ of the wave packet decreases. But simultaneously the uncertainty $(\Delta p)$ in momentum increases as $\Delta p \propto \Delta\lambda$. When $\Delta\lambda = \infty$, $\Delta p = \infty$ and $\Delta x = 0$, i.e, the wave pocket reduces to a point. In this position, there is no uncertainty in the position of the particle.

Now for a definite $\lambda = 0, \Delta\lambda = 0$, then $\Delta p = 0\ (\because \Delta p \propto \Delta\lambda)$, then there is no uncertainty in the momentum of the particle but in this case $\Delta x = \infty$, i.e., the position uncertainty is infinite since the associated wave with the particle may now extend up to $\infty$.

**Statement of the uncertainty principle:**

The uncertainty principle states that, it is fundamentally impossible to determine sumultaneously the position and the momentum of a particle to an accuracy better than one quantum of action h.

The mathematical expression of the particle is,

$$\Delta x \Delta p_x \geq \hbar,$$

where, $\hbar = \frac{h}{2\pi} = 1.0545 \times 10^{-31} Js$, h being the Planck's constant $= 6.62 \times 10^{-34} Js$. Here $\Delta x$ is the error (uncertainty) in the determination of position of the particle and $\Delta p_x$ is the error (uncertainty) in the determination of x-component of momentum. If the position be determined accurately, then $\Delta x$ becomes smaller and consequently larger is $\Delta p_x$ and vice versa. It is to be kept in mind that the uncertainties are not in the appratus used for the determination but in nature itself.

In three dimensions the uncertainty relations are expressed as follows:

$$\Delta x \Delta p_x \geq \hbar; \quad \Delta y \Delta p_y \geq \hbar; \quad \Delta z \Delta p_z \geq \hbar$$

(a) If $\Delta x = 0$, then, $\Delta p_x = \frac{\hbar}{\Delta x} \Rightarrow \Delta p_x = \frac{\hbar}{0} = \infty$.

This means that, if the uncertainty in the position measurement of a moving particle be zero, then at the same moment the uncertainty in the momentum measurement will be infinity.

(b) If the momentum of a moving particle at any instant be determined accurately (i.e., $\Delta p_x = 0$), then the uncertainty (error) in the determination of position of the particle will be infinite.

(c) For a particle of mass 'm' moving with velocity $v_x$ along the x-direction, its momentum is $p_x = mv_x$,

$$\Delta x \Delta p_x = \hbar, \quad \Delta x \Delta v_x = \frac{\hbar}{m}$$

For a heavy particle $m \to \infty$ and in that case, $\Delta x \Delta v_x = \frac{\hbar}{\infty} = 0$

For such a particle both the position and momentum can be determined accurately. This is true for macroscopic bodies whose motions are discussed in classical mechanics. Thus the uncertainty relation has no meaning in classical mechanics. But for microscopic objects such as electron, proton, atom, etc, quantum mechanics is applicable and as such the uncertainty relation is a reality in describing the dynamical behaviour of such objects.

**Time-Energy Uncertainty Relation:**

The time-energy uncertainty principle is stated as follows:

In any simultaneous determination of time and energy of a moving particle, the product of the uncertainties is equal to the Planck's constant, $\Delta E \Delta t \geq \hbar$

Here, $\Delta E$ is the uncertainty in the measurement of energy of $\Delta t$, the corresponding uncertainty in the measurement of time.

If $\Delta E$ be the maximum uncertainty in the determination energy of a system in a particular state, then $\Delta t$ is the minimum, i.e., the system remains for a minimum time in the state. On the other hand, if a system remains in a particular state for a maximum interval of time, then the uncertainty in the measurement of energy is the minimum.

### Angular momentum- angular position Uncertainty Relation:

$$\Delta L.\Delta\phi \geq \hbar$$

Where, $\Delta L$ is the uncertainty in the angular momentum of the particle and $\Delta\phi$ is the uncertainty in the angular position of the particle.

### Causality in Quantum Mechanics:

According to the principle of causality, knowledge of the behaviour of a system at one instant is sufficient for the prediction of its future behaviour. Thus, it is the quantum analogue of the classical principle that, the future behaviour of a particle can be determined if its position and velocity are known at a given instant. The law of causality, however, holds strictly in classical mechanics. As no observation can be made without disturbing a microsystem, it does not hold good for atomic systems.

According to Max Born, the principle of causality can be viewed from two alternate angles. In wave-particle picture the principle of causality breaks down. But if the system be described by a wave function $\psi(\vec{r},t)$ which satisfies a differential equation of first order in time, then the behaviour of the system at a point $P(\vec{r})$ at any time 't' can be described uniquely if $\psi$ is known at t = 0 at the same point $P(\vec{r})$. This shows that even for microsystems, the principle of causality is true. But the measurements on a microsystem cannot be carried out without seriously disturbing it. Hence it is impossible to determine the state of the system unambiguousy at t = 0. Hence it is impossible to determine the state of the system unambiguously at t = 0. Thus the principle of causality breaks down. Therefore, the probabilistic interpretation of the observed events are to be adopted in accordance with the uncertainty principle.

### Concept of Bohr Orbit in view of uncertainty principle:

We have the energy-time uncertainty relation, $\Delta E.\Delta t \geq \hbar$

According to the concept of Bohr, each electron revolves around the nucleus in quantised stationary orbits with definite energy E. So, the uncertainty is $\Delta E = 0$.

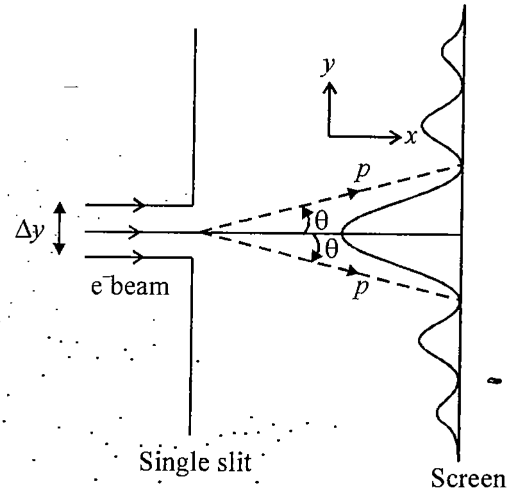
So, $$\Delta t = \frac{\hbar}{\Delta E} = \frac{\hbar}{0} = \infty$$

Hence, all energy states of the Bohr atom must have infinite lifetime $(\tau = \infty)$. This implied that radiation would not occur due to the transition of electrons. But actually $\Delta t \sim 10^{-8}$ sec leading to the occurrence of emission of radiation due to electronic transition. Thus the concept of Bohr orbit violets the Heisenberg's uncertainty principle.

Again, if $\Delta t$ is finite, then $\Delta E = \frac{\hbar}{\Delta t}$ is also finite. This means that the energy levels in the excited state must have a finite spread. As a result the radiation emitted due to the transition of an electron from the excited state to the ground state is not monochromatic. This leads to the conclusion that the spectral lines can never be infinitely sharp but must have a natural broadening. This is contradiction with Bohr's theory of spectral series.

### Electron difraction through a single slit:

Let a monoenergetic parallel beam of electrons moving along the x-direction be incident on a single split of width $\Delta y$. As the electron in motion is associated with de-Broglie wave, the electron beam after passing through the slit undergoes diffraction producing a fraunhofer diffraction pattern on the photoplate perpendicular to the x-axis.



The first minimum of the diffraction pattern satisfies the relation $\Delta y \sin\theta = \lambda$, where $\theta$ is the angle of diffraction for the first order and $\lambda$ is the de-Broglie wavelength. Now the uncertainty in the position of an electron in the slit is $\Delta y$ as it is very difficult to locte exactly the position of the electron in the slit,

$$\Delta y = \frac{\lambda}{\sin\theta}$$

Before incidence, let the momentum of the electron along the x-direction be p. After diffraction if the electron moves along OC, then the component of momentum p along the y-direction is $p\sin\theta$. Since the electron may be anywhere within $-\theta$ to $+\theta$, the y-component of momentum must be anywhere between $-p\sin\theta$ to $p\sin\theta$.

Thus the uncertainty in the momentum measurement of the electron is,

$$\Delta p_y = p\sin\theta - (-p\sin\theta) = 2p\sin\theta = 2\frac{h}{\lambda}\sin\theta$$

Therefore, $$\Delta y \cdot \Delta py = \frac{\lambda}{\sin\theta} \times \frac{2h}{\lambda}\sin\theta = 2h \geq h$$

To decrease the uncertainty in the determination position of the electron, $\Delta y$ must be decreased. But the above equation shows that a decrease in $\Delta y$ leads to the increase in $\theta$ and the diffraction pattern will be wider. This will cause a larger uncertainty in $\Delta p$. Thus the electron diffraction experiment established the truth of uncertainty principle.

**Consequences of Heisenberg's Uncertainty principle:**

**(1) Non-existance of electrons within the nucleus:**

Suppose electron exist within the nucleus, then uncertainty in the position of nucleus will be of the order of twice of radius of the nucleus i.e. $\Delta x = 2\times10^{-15}\,m$.

Then the minimum energy of the electron is given by

$$E_{\min}^2 = p_{\min}^2 c^2 + m_0^2 c^4 \Rightarrow E_{\min} = \sqrt{(\Delta p)^2 c^2 + m_0^2 c^4}$$

$$\Rightarrow E_{\min} = \sqrt{\left(\frac{\hbar}{2\Delta x}\right)^2 c^2 + m_0^2 c^4} \approx 10 MeV$$

Therefore, if electron exist within the nucleus, then minimum energy of the electron will be approximately 10 MeV. But, experimentally energy comes out to be of the order of 4 MeV. Thus, we can say that electron does not exist within the nucleus.

(2) **Ground state energy of a linear harmonic oscillator:**

Let a particle of mass m is executing SHM along x-axis and let at any time 't' its position is 'x' from the mean position. Therefore, $\Delta x = x$

Total energy of LHO $= E = E_k + E_P = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \approx \frac{(\Delta p)^2}{2m} + \frac{1}{2}m\omega^2(\Delta x)^2$

$$= \left(\frac{\hbar}{2x}\right)^2 \frac{1}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{\hbar^2}{8mx^2} + \frac{1}{2}m\omega^2 x^2$$

For minimum energy, $\frac{dE}{dx} = -\frac{\hbar^2}{4mx^3} + m\omega^2 x = 0 \Rightarrow x^2 = \frac{\hbar^2}{2m\omega}$

So, the minimum energy of the oscillator $= E_{\text{min.}} = \frac{\hbar^2}{8m}\frac{2m\omega}{\hbar} + \frac{1}{2}m\omega^2 \frac{\hbar^2}{2m\omega} = \frac{1}{2}\hbar\omega$

## SOLVED PROBLEMS

1. What voltage must be applied to an electron microscope to produce electrons of wavelength 0.40Å?

**Soln.** De-Broglie wavelength of an electron of mass m, charge 'e' and moving under a potential of V volts is,

$$\lambda = \frac{h}{\sqrt{2meV}}$$

From the above relation, $V = 936.3$ volts

2. What is the de-Broglie wavelength of an electron moving with velocity, $v = \frac{3}{5}c$?

**Soln.** The de-Broglie wavelength of a particle of mass 'm' moving with velocity 'v' is

$$\lambda = \frac{h}{mv} = \frac{h}{m_0 v}\sqrt{1 - v^2/c^2} = 0.0323\ A^0$$

3. You have 10 eV photon and 10 eV electron. Which one has shorter wave-length?

**Soln.** Wavelength of photon $\lambda = \frac{h}{p} = \frac{hc}{E} = \frac{6.62\times10^{-34}\times3\times10^8}{10\times1.6\times10^{-19}} = 1240Å$

De-Broglie wavelength of the electron

$$\lambda = \frac{h}{m_0 v} = \frac{h}{\sqrt{2m_0 E}} = 3.88\text{A}^0$$

The de-Broglie wavelength $\lambda$ of a particle is $\lambda \propto \frac{1}{\sqrt{E}}$ and for a photon $\lambda \propto \frac{1}{E}$. This shows that de-Broglie wavelength of an electron is much smaller as compared to that of a photon for the same energy E.

4. Energy of a particle at absolute temperature T is of the order of kT. Calculate the wavelength of thermal neutrons at 27°C. [Given, the mass of the neutron $= 1.67\times10^{-24}\ gm$. Planck's constant, $k = 6.67\times10^{-27}\ erg\sec$ and Boltzman constang, $k = 8.6\times10^{-5}\ eVk$]

**Soln.** De-Broglie wavelength of the neutron is

$$\lambda = \frac{h}{\sqrt{2mE}} = \frac{h}{\sqrt{2mkT}} = 1.78\text{Å}$$

5. A beam of monenergetic neutrons corresponding to 27°C is allowed to fall on a crystal. A first order reflection is observed at a glancing angle 30°. Calculate the interplanar spacing of the crystal.

Given, Planck constant, $h = 6.62 \times 10^{-34}\,Js$

Mass of neutron, $m = 1.67 \times 10^{-27}\,kg$

Boltzmann constant, $k = 1.38 \times 10^{-23}\,J/K$

**Soln.** From Bragg law of diffraction we have,

$$2d\sin\theta = n\lambda$$

where, the glancing angle, $\theta = 30°$, order number of diffraction, n = 1

$$\therefore \quad 2d\sin 30° = \lambda,\ or,\ 2d \times \frac{1}{2} = \lambda,\ or,\ \lambda = d$$

De-Broglie wavelength of the neutron is

$$\lambda = \frac{h}{\sqrt{2mE}} = \frac{h}{\sqrt{2mkT}} = 1.78\text{Å}.$$

Therefore, the interplanar spacing is 1.78 Å.

6. What is the de-Broglie wavelength of electrons of energy 0.1 MeV using both relativistic and non-relativistic expressions for $\lambda$? What is the percentage error in using non-relativistic expression for $\lambda$?

**Soln.** The non-relativistic de-Broglie wavelength of an electron of mass 'm' and energy eV is,

$$\lambda_{nr} = \frac{h}{\sqrt{2meV}} = \frac{12.26}{\sqrt{V}} = \frac{12.26}{\sqrt{0.1 \times 10^6}} = 0.0388\text{Å}$$

The relativistic expression for the de-Broglie wavelength of the same electron is,

$$\lambda_r = \frac{h}{\sqrt{2meV\left(1 + \frac{eV}{2mc^2}\right)}} = 0.0371\text{Å}$$

The percentage error in using non-relativistic expression, is

$$\frac{\lambda_{nr} - \lambda_r}{\lambda_{nr}} \times 100\% = \frac{(0.0388 - 0.0371)}{0.0388} \times 100\% = 4.38\%$$

7. Assume that the uncertainty in the location of a particle is equal to its de-Broglie wavelength. Calculate the uncertainty in the velocity of the particle.

**Soln.** According to the uncertainty principle,

$\Delta x \Delta p_x = \hbar$ [where $\Delta x$ and $\Delta p_x$ are the uncertainties in position and momentum]

$$\Rightarrow \Delta v_x = \frac{\hbar}{m\Delta x} = \frac{\hbar}{m\lambda} \quad \left[\because \Delta x = \lambda = \frac{h}{mv}\right]$$

$$= \frac{\hbar}{m \times \frac{h}{mv}} = \frac{v}{2\pi}$$

. Suppose that the momentum of a certain particle can be measured to an accuracy of one part in 1000. Determine the minimum uncertainty in the position of the particle if the particle is (i) a $5\times10^{-3}\,kg$ mass moving with a speed of 2 m/s; (ii) an elecron moving with a speed of $1.8\times10^{8}\ m/s$

**oln.** Given: $\frac{\Delta p}{p}=10^{-3} \Rightarrow \Delta p=10^{-3}p=10^{-3}mv$

Therefore, the minimum uncertainty in position of the particle will be

$$(\Delta x)_{\min}=\frac{h}{4\pi mv\times10^{-3}}$$

(i) $(\Delta x)_{\min}=\frac{h}{4\pi m_0 v\times10^{-3}}=5.28\times10^{-20}\,\text{Å}$

(ii) $(\Delta x)_{\min}=\frac{h\sqrt{1-v^2/c^2}}{4\pi m_0 v\times10^{-3}}=2.57\text{Å}$

). The average time interval that elapses between the excitation of an atom and the time it radiates energy is $10^{-8}$ sec. Calculate the uncertainty in the energy of emitted photons and the limit of accuracy with which the frequency of the emitted radiation may be determined.

**Soln.** Uncertainty in energy of the emitted photons will be

$$\Delta E=\frac{h}{\Delta t}=4.14\times10^{-7}eV$$

limit of accuracy with which the frequency of the emitted radiation may be determined will be

$$\Delta v=\frac{\Delta E}{h}=\frac{6.62\times10^{-26}}{1.6\times10^{-34}}=10^{8}\ cycle/\sec$$

Hence, the limit of accuracy with which the frequency of the emitted radiation may be determined is $\Delta v=10^{8}\,Hz$.

10. Find the average time that the atomic system remains in the energy state whose spectral line width is $10^{-4}\,\AA$ for $\lambda=4000\,\AA$.

**Soln.** From energy-time uncertainty relation, the average lifetime of the state will be

$$\Delta t=\frac{h}{4\pi\Delta E}=\frac{h}{4\pi\left(\frac{hc}{\lambda^2}\Delta\lambda\right)}=\frac{\lambda^2}{4\pi c\Delta\lambda}=4.24\ ns$$

11. In a Stern-Gerlach experiment, a collimated beam silver atom is produced by vapourising silver salt in a furnace at 1200°C. If the beam travels 100cm, use uncertainty principle to find the order of magnitude of the smallest spot that can be recorded on the detector.

[Given: Planck's constant, $h=6.6\times10^{-34}\,Js$; Boltzmann constant, $k=1.4\times10^{-23}\ J/K$. Mass of hydrogen atom $=1.66\times10^{-27}\,kg$]

**Soln.** If the silver atom of mass 'm' moves with velocity 'v' along the x-axis from the furnace at temperature T, then its kinetic energy is,

$$E_K=\frac{1}{2}mv^2=\frac{3}{2}kT. \qquad \therefore v=\sqrt{\frac{3kT}{m}}$$

Time required by the silver atom to travel a distance of 100 cm is,

$$t=\frac{100}{v}=\frac{100}{\sqrt{3kT/m}}$$

Now time-energy uncertainty relation is,

$$\Delta E\Delta t\approx\hbar\ \Rightarrow(\Delta E)_{\min}(\Delta t)_{\max}\approx\hbar\quad\Rightarrow(\Delta E)_{\min}\approx\frac{\hbar}{(\Delta t)_{\max}}=\frac{\hbar}{t}$$

Now, $$(\Delta E_{\min})^2=\frac{(\Delta p_{\min})^2}{2m}\Rightarrow\Delta p_{\min}=\sqrt{2mE_{\min}}=\sqrt{\frac{2m\hbar}{t}}$$

Maximum uncertainty in the position of the particle

$$(\Delta x)_{\max}\approx\frac{\hbar}{(\Delta p)_{\min}}=\sqrt{\frac{\hbar t}{2m}}=\sqrt{\frac{\hbar}{2m}}\frac{10}{\left(\frac{3kT}{m}\right)^{1/4}}=7\times10^{-7}\,m=0.7\,\mu m$$

The smallest spot on the detector corresponds to the maximum uncertainty in position of the silver atom. Hence, size of the smallest spot = 0.7 μm.

12. The speedometer of a 180 kg vehicle reads 72 km/h at a particular instant. The readings are known to have an accuracy of ±2 km/h. What would be the minimum uncertainty in the position of the vehicle at that instant according to the Heisenberg uncertainty perinciple?

**Soln.** The uncertainty in the momentum is

$$\Delta p_x=\Delta(mv_x)=m\Delta v_x=(180kg)\times(2km/hr)=100\,kg.m/s$$

Thus the minimum uncertainty in position is

$$(\Delta x)_{\min}=\frac{\hbar}{2\Delta p_x}=0.5\times10^{-36}\,m$$

Remember the size of an atom is of the order of $10^{-10}$ m and that of a nucleus is $10^{-15}$ m. A size of $10^{-36}$ m is too small to detect with our present day technology and hence we can safely ignore it.

13. Gamma rays of energy $E_0=14.4\,keV$ are emitted when $^{57}$Fe nuclei make transition from the first excited state to the ground state. The lifetime of this excited state is about $3\times10^{-8}$s. If the spread in the energy of the excited state is written as $\Delta E=\frac{v}{c}E_0$ find the value of v. Here c is the speed of light in vacuum.

**Soln.** From energy-time uncertainity principle,

$$\Delta E\Delta t\approx\hbar\ .\ \Rightarrow\Delta E\approx\frac{\hbar}{\Delta t}\ \Rightarrow\frac{v}{c}E_0\approx\frac{\hbar}{\Delta t}$$

$$\Rightarrow v\approx\frac{\hbar c}{E_0\tau}=\frac{hc}{2\pi E_0\tau}=\frac{1240\,eV\,nm}{2\pi\times14.4\times10^3\,eV}\approx0.12\,mm/s$$ nature.

14. A proton and an alpha particle have the same kinetic energy, then the relation between their de Broglie wavelength is **[BHU 2010]**

(a) $\lambda_p=\frac{\lambda_a}{2}$ (b) $\lambda_p=\frac{\lambda_a}{4}$ (c) $\lambda_p=2\lambda_a$ (d) $\lambda_p=4\lambda_a$

**Soln.** The ratio of de-Broglie wavelengths of a proton and an alpha particle is

$$\frac{\lambda_p}{\lambda_a}=\frac{h/\sqrt{2m_pK_p}}{h/\sqrt{2m_aK_a}}$$

where $K_p = K_a$ (given)

$$\therefore \quad \frac{\lambda_p}{\lambda_a} = \sqrt{\frac{m_a}{m_p}}$$

We know that, $m_a = 4\,m_p$

$$\therefore \quad \frac{\lambda_p}{\lambda_a} = \frac{2}{1} \Rightarrow \lambda_p = 2\lambda_a$$

**Correct option is (c)**

15. A neutron is confined within a nucleus of radius $5 \times 10^{-15}$ m. The minimum uncertainty in the momentum of nucleon [$m = 1.67 \times 10^{-27}$ kg, $\hbar = 1.05 \times 10^{-34}$ J sec] is **[BHU 2010]**

(a) $1.25 \times 10^{-15}$ kg-m/sec (b) $4.12 \times 10^{-18}$ kg-m/sec

(c) $1.05 \times 10^{-28}$ kg-m/sec (d) $1.65 \times 10^{-21}$ kg-m/sec

**Soln.** The minimum uncertainity in the momentum is $\Delta p = \frac{\hbar}{2\Delta r} = \frac{1.05 \times 10^{-34}}{2 \times 5 \times 10^{-15}} = 1.05 \times 10^{-20}$ kg-m/sec.

**Correct option is (c)**

16. If the uncertainty in the location of a particle is equal to its de-Broglie wavelength,then the uncertainty in its velocity $V$ is **[BHU 2011]**

(a) $\frac{V}{2}$ (b) $V$ (c) $\frac{V}{4}$ (d) $\frac{V}{3}$

**Soln.** $\Delta x \Delta p \approx \hbar \Rightarrow \Delta v \approx \frac{\hbar}{m\,\Delta x} = \frac{h}{m\frac{h}{mV}} = \frac{V}{2\pi}$

17. An electron has a speed of 300 metres/sec accurate up to 0.01%. With what fundamental accuracy can locate the position of this electron ($m_e = 9.1 \times 10^{-31}$ kg, charge $e = 1.6 \times 10^{-19}$ coulomb and Planck's constant $h = 6.64 \times 10^{-34}$ joules / sec)? **[BHU 2011]**

(a) 2.4 cm (b) 1.2 cm (c) 1.8 cm (d) 3.6 cm

**Soln.** $\Delta x \Delta p \approx h \Rightarrow \Delta x \approx \frac{h}{m\,\Delta v} = \frac{6.64 \times 10^{-34}}{(9.1 \times 10^{-31}) \times (300 \times 0.01\%)} m = 2.4\ cm$

**Correct option (a)**

18. The dispersion law for a certain type of wave motion is $\omega = (c^2k^2 + m^2)^{1/2}$, where $\omega$ is the angular frequency, $k$ is the magnitude of the propagation vector and $c$ and $m$ are constants. The group velocity of these waves approaches **[BHU 2012]**

(a) infinity as $k \to 0$ and $c$ as $k \to \infty$ (b) $c$ as $k \to 0$ and zero as $k \to \infty$

(c) zero as $k \to 0$ and infinity as $k \to \infty$ (d) zero as $k \to 0$ and $c$ as $k \to \infty$

**Soln.** Given : $\omega = (c^2k^2 + m^2)^{1/2}$

Group velocity $v_g = \frac{d\omega}{dk} = \frac{2c^2k}{2(c^2k^2 + m^2)^{1/2}} = \frac{c^2k}{\sqrt{c^2k^2 + m^2}}$

So, when $k \to 0$, $v_g = 0$

Now, $v_g = \dfrac{c^2 k}{\sqrt{c^2 k^2 + m^2}} = \dfrac{c^2}{\sqrt{c^2 + m^2/k^2}}$

So, when $k \to \infty$, $v_g = c$

**Correct option is (d)**

19. The uncertainty relation holds for the following situation **[BHU 2012]**

(a) holds for particles only

(b) does not hold for microscopic particles

(c) holds for both microscopic and macroscopic particles

(d) depends on the nature of the particle

**Soln.** The uncertainity relation holds for all particles whether microscopic or macroscopic.

**Correct option is (c)**

20. If the Broglie wavelength of an electron and proton are equal, then **[BHU 2012]**

(a) the velocity of proton is less than that of the electron

(b) the velocities of proton and electron are equal

(c) the velocity of proton is greater than that of the electron

(d) the energy of electron is less than that of the proton

**Soln.** Given : $\lambda_p = \lambda_e$

$$\Rightarrow \frac{h}{m_p v_p} = \frac{h}{m_e v_e} \Rightarrow \frac{v_e}{v_p} = \frac{m_p}{m_e} > 1 \Rightarrow v_e > v_p$$

**Correct option is (a)**

21. The position and momentum of a 1 keV electron are simultaneously determined. If its position is located to within 1 Å, what is the percentage of uncertainty in its momentum? **[BHU 2012]**

(a) 0.61 (b) 6.17 (c) 61.73 (d) 0.061

**Soln.** Given : $E = 1\,\text{keV}$

$$\Rightarrow \quad p = \sqrt{2mE} = \sqrt{2 \times 9.1 \times 10^{-31} \times 10^3 \times 1.6 \times 10^{-19}}$$

$$= 1.70 \times 10^{-24}\ \text{kg-m/s.}$$

The minimum uncertainity in the momentum is

$$\Delta p = \frac{\hbar}{\Delta x} = \frac{6.6 \times 10^{-34}}{2\pi \times 10^{-10}} = 10.5 \times 10^{-25}\ \text{kg-m/s.}$$

So the percentage in uncertainity of the momentum is

$$\frac{\Delta p}{p} \times 100 = \frac{10.5 \times 10^{-25}}{1.70 \times 10^{-24}} \times 100 \simeq 61.76$$

**Correct option is (c)**

22. A quantum particle in a three-dimensional space with momenta $\vec{p} = (p_x, p_y, p_z)$ and position $\vec{r} = (x, y, z)$. The uncertainty in the measurement of position and momentum along $z$-axis is **[BHU 2013]**

(a) $\Delta z\, \Delta p_y \geq \hbar$ (b) $\Delta y\, \Delta p_z \geq \hbar$ (c) $\Delta x\, \Delta p_z \geq \hbar$ (d) $\Delta z\, \Delta p_z \geq \hbar$

**Soln.** The uncertainity relation is given by $\Delta x_i\ \Delta p_i \geq \hbar$.

So for $i = 3$, i.e., $x_3 = z$ and $p_3 = p_z$, we have $\Delta z\ \Delta p_z \geq \hbar$.

**Correct option is (d)**

3. An electron is confined to a box of length $10^{-8}$ m. Calculation of the minimum uncertainty in its velocity, with $m_e = 9\times10^{-31}$ kg, $\hbar = 1.05\times10^{-34}$ J-sec is **[BHU 2013]**

(a) $1.17\times10^4$ m/sec (b) $1.17\times10^6$ m/sec (c) $1.17\times10^2$ m/sec (d) $1.17\times10^8$ m/sec

**oln.** Given : $\Delta x = 10^{-8}$ m

Using, $\Delta x\,\Delta p_x \geq \hbar$ **or** $m\Delta x\,\Delta v_x \geq \hbar$ **or** $\Delta v_x \geq \dfrac{\hbar}{m\Delta x}$ **or** $\Delta v_x \geq \dfrac{1.05\times10^{-34}}{9.1\times10^{-31}\times10^{-8}}$

**or** $\Delta v_x \geq 0.115\times10^{-15}$ **or** $\Delta v_x \geq 1.15\times10^4$

So $(\Delta v_x)_{\min} \approx 1.15\times10^4$ m/s.

**Correct option is (a)**

4. An electron is confined to a box of length $L$. If the length of the box changes to $2L$, how would the uncertainty the uncertainty of momentum charge ? **[BHU 2014]**

(a) uncertainty of momentum will be twice (b) uncertainty of momentum will be half

(c) uncertainty of momentum will be one fourth (d) uncertainty of momentum will be four times

**oln.** The uncertainity of momentum in particle in a box problem is $\Delta p = \dfrac{n\pi\hbar}{L}$

So if the length of the box changes to $2L$, the uncertainity in momentum gets half.

**Correct option is (b)**

5. The de-Broglie wavelength of an electron with energy 100 eV is equal to : **[BHU 2014]**

(a) 12.3 Å (b) 1.23 Å (c) 123 Å (d) 0.123 Å

**oln.** The de-Broglie wavelength is

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE_k}}$$

$$= \frac{6.6\times10^{-34}}{\sqrt{2\times9.1\times10^{-31}\times100\times1.6\times10^{-19}}} \approx 1.22\times10^{-10} = 1.22\ \text{Å}$$

**Correct option is (b)**

26. The de Broglie wavelength, associated with a light of wavelength $(\lambda)$ is : **[BHU 2016]**

(a) $2\lambda$ (b) $\dfrac{3}{2}\lambda$ (c) $\dfrac{2}{3}\lambda$ (d) $\lambda$

**Soln.** The concept of de-Broglie wavelength is valid only for moving particles not for light. So, the wavelength of light of wavelength λ is remained to be λ.

**Correct option is (d)**

27. The de Broglie wavelength, associated with a neutron at temperature 27°C, is nearly equal to **[BHU 2016]**

(a) 2.77 Meter (b) 1.77 Angstrom (c) 2.77 Micrometer (d) 1.77 Nanometer

**Soln.** The thermal wavelength of a particle is given by $\lambda = \dfrac{h}{\sqrt{3m\,k_BT}}$, where $m$ is the mass of the particle and $k_B$ is the Boltzmann's constant.

$$\therefore\ \lambda = \frac{6.6\times10^{-34}}{(3\times1.67\times10^{-27}\times1.38\times10^{-23}\times300)^{1/2}} \simeq 1.45\ \text{Å}.$$

**Correct option is (b)**

28. The de Broglie wavelengths associated with 2eV photon and electron are : **[BHU 2016]**

(a) $\approx 6200$ Å and $\approx 20$ Å, respectively (b) $\approx 7200$ Å and $\approx 9$ Å, respectively

(c) $\approx 6200$ Å and $\approx 9$ Å, respectively (d) $\approx 7200$ Å and $\approx 20$ Å, respectively

**Soln.** For electron, de-Broglie wavelength is

$$\lambda = \frac{hc}{E} = \frac{6.6\times10^{-34}\times3\times10^{8}}{2\times1.6\times10^{-19}} \Rightarrow \lambda \simeq 6.18\times10^{-7} \simeq 6200\text{ Å}.$$

Now, for electron, the de-Broglie wavelength is

$$\lambda = \frac{h}{\sqrt{2mE}} = \frac{6.6\times10^{-34}}{\sqrt{2\times9.1\times10^{-31}\times2\times1.6\times10^{-9}}} \simeq 0.86\times10^{-19} \simeq 9\text{ Å}.$$

**Correct option is (c)**

29. An electron of mass $m$ and charge $e$, initially at rest, gets accelerated by a constant electric field $E$. The rate of change of de-Broglie wavelength of this electron with time, (ignoring relativistic effects) is given by

(a) $-\frac{h}{eEt^2}$ (b) $-\frac{eht}{E}$ (c) $-\frac{mh}{eEt^2}$ (d) $-\frac{h}{eE}$ **[HCU 2010]**

**Soln.** The de-Broglie wavelength is $\lambda = \frac{h}{p}$

Therefore, $\frac{d\lambda}{dt} = -\frac{h}{p^2}\frac{dp}{dt}$, where $\frac{dp}{dt} = eE$ and $p = eEt$

Therefore, $\frac{d\lambda}{dt} = -\frac{h}{eEt^2}$.

**Correct option is (a)**

30. The uncertainty relation holds for the following situation **[HCU 2010]**

(a) holds for particles only

(b) does not hold for macroscopic particles

(c) holds for both microscopic and macroscopic particles

(d) depends on the nature of the particle

oln. The uncertainity relation holds for all particles whether they are microscopic or macroscopic.

**Correct option is (c)**

1. The phase velocity of pressure waves $v_p$ is related to the wave vector $k$ by $v_p = Ck^{-1/2}$. ($C$ is a constant). The group velocity is given by **[HCU 2010]**

(a) $\frac{v_p}{2}$ (b) $2v_p$ (c) $kv_p$ (d) zero

oln. Phase velocity $v_p = \frac{\omega}{k} = \frac{C}{\sqrt{k}} \Rightarrow \omega = C\sqrt{k}$

Group velocity $v_g = \frac{d\omega}{dk} = \frac{C}{2\sqrt{k}} = \frac{v_p}{2}$

**Correct option is (a)**

2. The de-Broglie wavelength of an electron with a kinetic energy of 120 eV is nearly **[HCU 2010]**

(a) 1000 pm (b) 100 pm (c) 10 pm (d) 1 pm

**oln.** The de-Broglie wavelength is

$$\lambda = \frac{h}{\sqrt{2mE_k}} = \frac{6.6\times10^{-34}}{\sqrt{2\times9.1\times10^{-31}\times120\times1.6\times10^{-19}}} \simeq 100 \text{ pm}.$$

**Correct option is (b)**

3. The de-Broglie wavelength of a neutron (kinetic energy 3/2 kT) at 0 °K is
(a) 1.5 Å (b) 2 Å (c) 0 Å (d) 273 Å **[HCU 2011]**

**oln.** The de-Broglie wavelength is $\lambda = \frac{h}{\sqrt{2mE_k}}$, where $E_k = \frac{3}{2}kT$

$$\therefore\ \lambda = \frac{h}{\sqrt{3mkT}} = \frac{6.6\times10^{-34}}{\sqrt{3\times1.67\times10^{-27}\times1.38\times10^{-23}\times273}} \simeq 1.5\ \text{Å}.$$

**Correct option is (a)**

4. If surface waves in a lake have a phase velocity given by $v = a\sqrt{\lambda/2\pi}$, where $a$ is constant, then their group velocity is **[HCU 2012]**

(a) $v$ (b) $2v$ (c) $\frac{v}{2}$ (d) $\frac{v}{a}$

**oln.** Group velocity $v_g = v_p - \lambda\frac{dv_p}{d\lambda} = a\sqrt{\frac{\lambda}{2\pi}} - \lambda a\frac{1}{\sqrt{2\pi}}\frac{1}{2\sqrt{\lambda}} = \frac{a}{2}\sqrt{\frac{\lambda}{2\pi}} = \frac{v}{2}$

**Correct option is (c)**

5. De-Broglie hypothesized that the linear momentum and wavelength of a free massive particle are related to one of the following constants. **[HCU 2014]**

(a) Rydberg's constant (b) Speed of light

(c) Planck's constant (d) Boltzmann's constant

**oln.** According to de-Broglie hypothesized $\lambda = \frac{h}{p}$; (where λ is planck's constant).

**Correct option is (c)**

36. The de-Broglie wavelength of electrons $\left(m_e = 9.1\times10^{-31}kg \text{ and } \in = 1.6\times10^{-19}C\right)$ accelerated from rest through a potential difference of V (volts) is **[HCU 2016]**

(a) $\sqrt{\frac{15}{V}}\,nm$ (b) $\sqrt{\frac{1.5}{V}}\,nm$ (c) $\frac{15}{\sqrt{V}}\,nm$ (d) $\sqrt{\frac{V}{1.5}}\,nm$

**Soln.** According to energy conservation $\frac{1}{2}mv_e^1 = \text{eV} \Rightarrow p_e = \sqrt{2m_e\,\text{eV}}$

So the de-Broglie wavelength of electrons is

$$\lambda_e = \frac{h}{p_e} = \frac{h}{\sqrt{2m_e\,\text{eV}}} = \frac{6.62\times10^{-34}}{\sqrt{2\times9.1\times10^{-31}\times1.6\times10^{-19}\times \text{V}}} = \sqrt{\frac{1.5}{\text{V}}}\ \text{nm}.$$

**Correct option is (b)**

# Chapter 4

# Wave Functions & Operators

## 4.1 Introduction :

There is always a quantity asscociated with any type of waves, which varies periodically with space and time. In water waves, the quantity that varies periodically is the height of the water surface and in light waves, electric field and magnetic field vary with space and time. For De-broglie waves or matter waves asscociated with a moving particle, the quantity that vary with space and time, is called wave function of the particle.

The temporal and spatial evolution of a quantum mechanical particle is described by a wave function $\psi(x,t)$ for 1-D motion and $\psi(\vec{r},t)$ for 3-D motion. . It contains all possible information about the state of the system. These are known as configuration space wave function.

**Condition for a physically accepted, well behaved, realistic wave function :**

(i) $\psi(x,t)$ should be finite, single-valued and continuous everywhere in space.

(ii) $\frac{d\psi}{dx}$ should be continuous everywhere in space. But, $\frac{d\psi}{dx}$ may be discontinuous in some cases as follows:

(a) If the potential under which the particle is moving, has an infinite amount of discontinuity at some points,
(b) If the potential under which the particle is moving, is of dirac delta nature.

(iii) $\psi(x,t)$ should be square integrable i.e. $\int_{-\infty}^{\infty}|\psi(x,t)|^2 dx$ = finite quantity

**Example 1:** Which of the following wave function is acceptable as the solution of the Schrodinger equation for all values of $x$?

(a) $\psi(x) = A\sec x$ (b) $\psi(x) = A\tan x$ (c) $\psi(x) = Ae^{x^2}$ (d) $\psi(x) = Ae^{-x^2}$

**Soln.** $\psi(x) = A\sec x$ and $\psi(x) = A\tan x$ is not finite at $x = \frac{\pi}{2}$

$\psi(x) = Ae^{x^2}$ is not finite at $x = \pm\infty$ ; $\psi(x) = Ae^{-x^2}$ is finite everywhere in space.

**Correct option is (d)**

**Example 2:** A ball bounces back off earth. You are asked to solve this quantum mechanically assuming the earth is an infinitely hard sphere. Consider surface of earth as the origin implying $V(0) = \infty$ and a linear potential eleswhere (i.e. $V(x) = -mgx \; for \; x > 0$). Which of the following wave function is physically admissible for this problem (with $k > 0$):

(a) $\psi(x) = \frac{e^{-kx}}{x}$ (b) $\psi(x) = Ae^{-kx^2}$ (c) $\psi(x) = -Axe^{kx}$ (d) $\psi(x) = Axe^{-kx^2}$

**Soln.** Since the earth has been taken as infinitely hard sphere, therefore the wave function of the ball will be zero for $x < 0$, non-zero for $x > 0$ and zero at infinity.

Since $\psi(x)$ should be continous everywhere, the wavefunction of the ball will be zero at $x = 0$.

**Correct option is (d).**

## 4.2 Physical significance of wave function :

Generally, $\psi(x,t)$ is a complex quantity. It can be multipliedd by any complex number without affecting its physical significance. In general, $\psi(x,t)$ has no direct physical significance. But the quantity $\psi^*(x,t)\psi(x,t) = |\psi(x,t)|^2$ is real, physically significant and is defined as position probability density i.e. probability of finding the particle per unit length at time '$t$'. Therefore, for 1-D motion the probability of finding the particle between $x$ to $x + dx$ at time '$t$' is given by

$$\psi^*(x,t)\psi(x,t)\,dx = |\psi(x,t)|^2\,dx$$

and for 3-D motion the probability of finding the particle within the volume element $d\tau$ located between $\vec{r}$ and $\vec{r} + d\vec{r}$ at time 't' is given by

$$\psi^*(\vec{r},t)\psi(\vec{r},t)\,d\tau = |\psi(\vec{r},t)|^2\,d\tau$$

Now, $\psi(x,t)$ or $\psi(\vec{r},t)$ should be chosen such that total probability of finding the particle in the entire space should be equal to unity i.e.

$$\int_{-\infty}^{\infty} \psi^*(x,t)\psi(\vec{r},t)\,d\tau = \int_{-\infty}^{\infty} |\psi(x,t)|^2\,d\tau = 1 \text{ (for 1-D motion)}$$

$$\int_{\substack{all\\space}} \psi^*(\vec{r},t)\psi(\vec{r},t)\,d\tau = \int_{\substack{all\\space}} |\psi(\vec{r},t)|^2\,d\tau = 1 \text{ (for 3-D motion)}$$

This is called the normalization condition of the wave function.

If $\psi(x,t)$ or $\psi(\vec{r},t)$ is normalized at some time it remains normalized forever. This can be understood as a conservation of probability or conservation of normalization.

**Method of Normalization:**

Consider $\varphi(x,t)$ is an unnormalized wave function. We can construct a normalized wave function as $\psi(x,t) = N\varphi(x,t)$ where $N$ is the normalization constant. Therefore,

$$\int_{-\infty}^{\infty} \psi^*(x,t)\psi(x,t)\,dx = |N^2| \int_{-\infty}^{\infty} \varphi^*(x,t)\varphi(x,t)dx = 1$$

$$\Rightarrow N = \frac{1}{\sqrt{\int_{-\infty}^{\infty} \varphi^*(x,t)\varphi(x,t)dx}}$$

**Example 3:** Normalize the wave function given by

$$\psi(x) = Ne^{-\alpha x} \quad (x > 0)$$
$$= Ne^{\alpha x} \quad (x < 0)$$

**Soln.** Normalization condition is

$$\int_{-\infty}^{\infty}\psi^*\psi dx = 1 \Rightarrow |N|^2\left[\int_{-\infty}^{0} e^{2\alpha x}dx + \int_{0}^{\infty} e^{-2\alpha x}dx\right] = 1$$

$$\Rightarrow |N|^2\left[\frac{1}{2\alpha}+\frac{1}{2\alpha}\right] = 1 \quad \Rightarrow N = \sqrt{\alpha}$$

**Orthogonality condition of wave functions :**

Two wavefunctions $\psi_m(x)$ and $\psi_n(x)$ are said to be orthogonal to each other, if

$$\int_{-\infty}^{\infty}\psi_m^*(x)\psi_n(x)dx = 0\ (m \neq n)$$

i.e. if a particle is in the state $\psi_m(x)$, then the particle cannot be in the state $\psi_n(x)$ simultaneously together.

**Orthonormality condition of wave functions :**

Two wavefunctions $\psi_m(x)$ and $\psi_n(x)$ are said to be orthonormal to each other, if

$$\int_{-\infty}^{\infty}\psi_m^*(x)\psi_n(x)dx = 0\ (m \neq n)$$

## 4.3 Operator formalism :

An operator is a mathematical rule (or procedure) which operating on one function transforms it into another function i.e. $\hat{A}\psi(x) = \phi(x)$. Every dynamical variable in Quantum mechanics represented by a operator.

| **Operations** | **Symbol** | **Result of the operation** |
|---|---|---|
| Taking the square root | $\sqrt{\ }$ | $\sqrt{x^m} = x^{m/2}$ |
| Differentiation w.r.t. x | $\frac{d}{dx}$ | $\frac{d}{dx}(x^m) = mx^{m-1}$ |

Position Operator : $\hat{x}, \hat{y}, \hat{z}$

Momentum Operator : $\hat{p}_x = -i\hbar\frac{\partial}{\partial x},\ \hat{p}_y = -i\hbar\frac{\partial}{\partial y},\ \hat{p}_z = -i\hbar\frac{\partial}{\partial z} \Rightarrow \hat{p} = -i\hbar\vec{\nabla}$

Potential energy Operator : $\hat{V}$

Kinetic energy Operator : $\hat{K} = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\vec{\nabla}^2$

**Commutator Bracket:**

The commutator bracket of two operators $\hat{A}$ and $\hat{B}$ is defined as $[\hat{A},\hat{B}] = \hat{A}\hat{B}-\hat{B}\hat{A}$.

Example:

(1) $\left[\hat{x},\frac{d}{dx}\right]\psi = \hat{x}\frac{d\psi}{dx}-\frac{d}{dx}(\hat{x}\psi) = x\frac{d\psi}{dx}-x\frac{d\psi}{dx}-\psi = -\psi \Rightarrow \left[\hat{x},\frac{d}{dx}\right] = -1$

(2) $[\hat{x},\hat{p}_x]\psi = \hat{x}\hat{p}_x\psi - \hat{p}_x\hat{x}\psi = -i\hbar x\dfrac{\partial\psi}{\partial x} + i\hbar\dfrac{\partial}{\partial x}(x\psi) = i\hbar\psi \Rightarrow [\hat{x},\hat{p}_x] = i\hbar$

Similarly, $\left[\hat{y},\hat{p}_y\right] = i\hbar$ and $\left[\hat{z},\hat{p}_z\right] = i\hbar$

**Linear Operator:**

If an operator $\hat{A}$ is said to be a linear if

(i) $\hat{A}\left[c\psi(x)\right] = c\hat{A}\psi(x)$ and (ii) $\hat{A}\left[\psi_1(x)+\psi_2(x)\right] = \hat{A}\psi_1(x)+\hat{A}\psi_2(x)$

All Quantum mechanical Operators are linear in nature.

**Properties of the commutator bracket:**

1. $\left[\hat{A},\hat{B}\right] = -\left[\hat{B},\hat{A}\right]$ 2. $\left[\hat{A},\hat{B}+\hat{C}+\hat{D}+....\right] = \left[\hat{A},\hat{B}\right]+\left[\hat{A},\hat{C}\right]+........$

3. $\left[\hat{A},\hat{B}\right]^\dagger = \left[\hat{B}^\dagger,\hat{A}^\dagger\right]$ 4. $\left[\hat{A},\hat{B}\hat{C}\right] = \left[\hat{A},\hat{B}\right]\hat{C}+\hat{B}\left[\hat{A},\hat{C}\right]$

5. $\left[\hat{A},\left[\hat{B}\hat{C}\right]\right]+\left[\hat{B},\left[\hat{C},\hat{A}\right]\right]+\left[\hat{C},\left[\hat{A},\hat{B}\right]\right] = 0$

6. $\left[\hat{A},f\left(\hat{A}\right)\right] = 0$ 7. $\left[f\left(\hat{A}\right),G\left(\hat{A}\right)\right] = 0$

8. $\left[f\left(\hat{A}\right),G\left(\hat{B}\right)\right] = 0$ only if $\left[\hat{A},\hat{B}\right] = 0$

9. $\left[\hat{A},\hat{B}^n\right] = n\hat{B}^{n-1}\left[\hat{A},\hat{B}\right]$ 10. $\left[\hat{A}^n,\hat{B}\right] = n\hat{A}^{n-1}\left[\hat{A},\hat{B}\right]$

**Example 4:** Find the commutator bracket $\left[\left[\hat{A},\hat{B}\right],\left[\hat{B},\hat{A}\right]\right]$

**oln:** $\left[\left[\hat{A},\hat{B}\right],\left[\hat{B},\hat{A}\right]\right] = \left[\hat{A},\hat{B}\right]\left[\hat{B},\hat{A}\right]-\left[\hat{B},\hat{A}\right]\left[\hat{A},\hat{B}\right]$

$$= \left(\hat{A}\hat{B}-\hat{B}\hat{A}\right)\left(\hat{B}\hat{A}-\hat{A}\hat{B}\right)-\left(\hat{B}\hat{A}-\hat{A}\hat{B}\right)\left(\hat{A}\hat{B}-\hat{B}\hat{A}\right)$$

$$= \hat{A}\hat{B}\hat{B}\hat{A}-\hat{A}\hat{B}\hat{A}\hat{B}-\hat{B}\hat{A}\hat{B}\hat{A}+\hat{B}\hat{A}\hat{A}\hat{B}-\hat{B}\hat{A}\hat{A}\hat{B}+\hat{A}\hat{B}\hat{A}\hat{B}+\hat{B}\hat{A}\hat{B}\hat{A}-\hat{A}\hat{B}\hat{B}\hat{A} = 0$$

**Example 5:** Find the commutator bracket $\left[x^2,p^2\right]$.

**oln:** $\left[x^2,p^2\right] = \left[x^2,pp\right] = \left[x^2,p\right]p+p\left[x^2,p\right] = 2x[x,p](p)+(p)2x[x,p] = 2i\hbar(xp+px)$

**Example 6:** If $\hat{H} = \dfrac{p^2}{2m}+V(x)$, then calculate $\left[x,\left[x,\hat{H}\right]\right]$.

**oln:** $\left[x,\hat{H}\right] = \left[x,\dfrac{p^2}{2m}\right]+\left[x,V(x)\right] = \dfrac{1}{2m}\left[x,p^2\right]+0 = \dfrac{1}{2m}2p[x,p] = \dfrac{i\hbar}{m}p$

$$\left[x,\left[x,\hat{H}\right]\right] = \left[x,\frac{i\hbar}{m}p\right] = \frac{i\hbar}{m}[x,p] = -\frac{\hbar^2}{m}$$

**Eigenvalues and Eigenfunctions:**

If an operator $\hat{\text{A}}$ operating on a function $\psi_n(x)$ gives $\hat{A}\psi_n(x) = \lambda\psi_n(x)$, then $\psi_n(x)$ is called the eigenfunction of $\hat{\text{A}}$ corresponding to eigenvalue $\lambda$ and the above equation is known as eigenvalue equation of operator $\hat{\text{A}}$.

Example: $\hat{A} = \frac{d^2}{dx^2}$ and $\psi_n(x) = ae^{-2x} \Rightarrow \hat{A}\psi_n(x) = \frac{d^2}{dx^2}\left(ae^{-2x}\right) = 4ae^{-2x} = 4\psi_n(x)$

$\psi_n(x)$ is an eigenfunction of $\hat{\text{A}}$ corresponding to eigenvalue 4.

If the commutator bracket of two operators $\hat{\text{A}}$ and $\hat{\text{B}}$ is equal to zero i.e. $\left[\hat{A}, \hat{B}\right] = 0$, then the physical observables corresponding to these operators are simultaneously accurately measurable and they have a complete set of simultaneous eigenfunctions.

**Example 7:** The operator $\left(x + \frac{d}{dx}\right)$ has the eigenvalue $\alpha$. Determine the corresponding wavefunction.

**Soln:** Eigenvalue equation: $\left(x + \frac{d}{dx}\right)\psi = \alpha\psi \Rightarrow \frac{d\psi}{dx} = (\alpha - x)\psi \Rightarrow \int \frac{d\psi}{\psi} = \int (\alpha - x)\,dx$

$$\Rightarrow \ln\psi = \left(\alpha x - \frac{x^2}{2}\right) + \ln\psi_0 \Rightarrow \psi = \psi_0 \exp\left(\alpha x - \frac{x^2}{2}\right)$$

**Example 8:** Find the constant B which makes $e^{-ax^2}$ an eigenfunction of the operator $\left(\frac{d^2}{dx^2} - Bx^2\right)$.

What is the corresponding eigenvalue?

**Soln:** $\left(\frac{d^2}{dx^2} - Bx^2\right)e^{-ax^2} = \left(4a^2x^2 - 2a + Bx^2\right)e^{-ax^2}$

For $e^{-ax^2}$ to be eigenfunction of the operator $\left(\frac{d^2}{dx^2} - Bx^2\right)$, then the eigenvalue

$\left(4a^2x^2 - 2a - Bx^2\right)$ must be independent of x i.e. $\left(4a^2 - B\right) = 0 \Rightarrow B = 4a^2$

Therefore, $\left(\frac{d^2}{dx^2} - Bx^2\right)e^{-ax^2} = -2ae^{-ax^2}$; Thus the eigenvalue of the operator is (-2a).

**Hermitian operator:**

A operator $\hat{\text{A}}$ is said to be hermitian if $\hat{A}^\dagger = \hat{A}$ and its should satisfy the following relation:

Dirac representation: $\left(\langle\phi|\hat{A}|\psi\rangle\right)^\dagger = \langle\psi|\hat{A}^\dagger|\phi\rangle = \langle\psi|\hat{A}|\phi\rangle$

Schrodinger representation: $\left(\phi, \hat{A}\psi\right) = \left(\hat{A}\phi, \psi\right) \Rightarrow \int \phi^* \hat{A}\psi\, d\tau = \int \left(\hat{A}\phi\right)^* \psi\, d\tau$

**Properties:**

(1) Eigenvalues of hermitian operators are real.

(2) Eigenfunctions corresponding to different eigenvalues of a hermitian operator are orthogonal.

(3) All physical observable in quantum mechanics are represented by hermitian operators.

Example: $\left(\phi, \hat{p}_x\psi\right) = \int_{-\infty}^{\infty} \phi^* \left(-i\hbar\frac{\partial}{\partial x}\right)\psi\, dx = \int_{-\infty}^{\infty} \phi^* \left(-i\hbar\frac{\partial\psi}{\partial x}\right)dx$

$$= -i\hbar\left[\phi^*\psi\right]_{-\infty}^{\infty} + i\hbar\int_{-\infty}^{\infty}\left(\frac{d\phi^*}{dx}\right)\psi\, dx \quad = -i\hbar\int_{-\infty}^{\infty}\left(\frac{d\phi^*}{dx}\right)\psi\, dx = \int_{-\infty}^{\infty} \hat{p}_x^* \phi^* \psi\, dx$$

Hence, the momentum operator $\hat{p}_x$ is hermitian in nature.

Similarly, we can see that $\frac{\partial}{\partial x}$ is anti-hermitian i.e. $\left[\frac{\partial}{\partial x}\right]^\dagger = -\frac{\partial}{\partial x}$

**Unitary Operator:**
An operator $\hat{A}$ is said to be unitary if $\hat{A}^\dagger = \hat{A}^{-1}$

**Example 9:** Let, $N = \hat{b}^\dagger\hat{b}$ where the operator $\hat{b}$ satisfies the relation $\hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b} = 1$ and $\hat{b}^2 = \left(\hat{b}^\dagger\right)^2 = 0$; then eigenvalues of N are

(a) +ve and –ve integers (b) all +ve integers (c) ±1 and 0 only (d) 0 and 1 only

**Soln:** $\hat{N}|\psi\rangle = \lambda|\psi\rangle \Rightarrow \hat{b}^\dagger\hat{b}|\psi\rangle = \lambda|\psi\rangle \Rightarrow \left(\hat{b}^\dagger\hat{b}\right)^2|\psi\rangle = \lambda^2|\psi\rangle$

$$\Rightarrow \hat{b}^\dagger\hat{b}\hat{b}^\dagger\hat{b}|\psi\rangle = \lambda^2|\psi\rangle \Rightarrow \hat{b}^\dagger\left(1-\hat{b}^\dagger\hat{b}\right)\hat{b}|\psi\rangle = \lambda^2|\psi\rangle$$

$$\Rightarrow \hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b}|\psi\rangle = \lambda^2|\psi\rangle \Rightarrow \hat{b}^\dagger\hat{b}|\psi\rangle = \lambda^2|\psi\rangle \Rightarrow \lambda|\psi\rangle = \lambda^2|\psi\rangle \Rightarrow \lambda = 0, 1$$

**Example 10:** Let the wave function of a particle of mass 'm' at a given instant be $\psi(x) = Ae^{-x^2/a^2}$. What will be the function $K\psi(x)$ where 'K' is the kinetic energy? Is this wave function an eigenfunction of kinetic energy?

**Soln.** The kinetic energy operator is $K = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$. Thus,

$$K\psi(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\left(Ae^{-x^2/a^2}\right) = -\frac{\hbar^2}{2m}\frac{d}{dx}\left[-\frac{2A}{a^2}xe^{-x^2/a^2}\right]$$

$$= -\frac{\hbar^2}{2m}\frac{d}{dx}\left[-\frac{2A}{a^2}xe^{-x^2/a^2}\right] = \frac{2A\hbar^2}{2ma^2}\left[x\left(\frac{-2x}{a^2}e^{-x^2/a^2}\right) + e^{-x^2/a^2}\right] = \frac{A\hbar^2}{ma^2}\left(1-\frac{2x^2}{a^2}\right)e^{-x^2/a^2}$$

This is not an eigenfunction of kinetic energy.

**Example 11:** An operator A is defined as $A\psi(x) = \psi^*(x)\psi(x)$. Is this a linear operator?

**Soln.** For the given operator,

$$A\left[\psi_1(x)+\psi_2(x)\right] = \left[\psi_1(x)+\psi_2(x)\right]^*\left[\psi_1(x)+\psi_2(x)\right]$$

$$= \left[\psi^*{}_1(x)+\psi^*{}_2(x)\right]\left[\psi_1(x)+\psi_2(x)\right]$$

$$= \psi^*{}_1(x)\psi_1(x)+\psi^*{}_2(x)\psi_2(x)+\psi^*{}_1(x)\psi_2(x)+\psi^*{}_2(x)\psi_1(x)$$

and $A\psi_1(x)+A\psi_2(x) = \psi^*{}_1(x)\psi_1(x)+\psi^*{}_2(x)\psi_1(x)$

Thus, $A\left[\psi_1(x)+_2(x)\right] = \psi^*{}_1(x)+A\psi_2(x)$ and hence A is not a linear operator.

**Example 12.** Prove that $[Xp_x, H] = \frac{i\hbar}{m}p_x^2 + X[p_x, V]$ where $H = \frac{p_x^2}{2m}+V$, and V is potential energy operator.

**Soln.** $[Xp_x, H] = X[p_x, H]+[X, H]p_x = X\left[p_x, \frac{p_x^2}{2m}+V\right]+\left[X, \frac{p_x^2}{2m}+V\right]p_x$

$$= X[p_x, V] + \frac{1}{2m}\{p_x[X, p_x] + [X, p_x]p_x\}p_x$$

$$= X[p_x, V] + \frac{1}{2m}(2i\hbar p_x)p_x = \frac{i\hbar}{m}p_x^2 + X[p_x, V]$$

**Example 13.** Prove that the operators $i(d/dx)$ and $d^2/dx^2$ are Hermitian.

**Soln.** $\int_{-\infty}^{\infty}\psi_m^*\left(i\frac{d}{dx}\right)\psi_n dx = i[\psi_m^*\psi_n]_{-\infty}^{\infty} - i\int_{-\infty}^{\infty}\psi_n\frac{d}{dx}\psi_m^* dx \quad = \int_{-\infty}^{\infty}\left(i\frac{d}{dx}\psi_m\right)^*\psi_n dx$

Therefore, $id/dx$ is Hermitian.

$$\int_{-\infty}^{\infty}\psi_m^*\frac{d^2\psi_n}{dx^2}dx = \left[\psi_m^*\frac{d\psi_n}{dx}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{d\psi_n}{dx}\frac{d\psi_m^*}{dx}dx$$

$$= \left[\frac{d\psi_m^*}{dx}\psi_n\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\psi_n\frac{d^2\psi_m^*}{dx^2}dx = \int_{-\infty}^{\infty}\frac{d^2\psi_m^*}{dx^2}\psi_n dx$$

Thus, $d^2/dx^2$ is Hermitian.

**Example 14:** Find the following commutation relations:

(i) $\left[\frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}\right]$ (ii) $\left[\frac{\partial}{\partial x}, F(x)\right]$

**Soln.** (i) $\left[\frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}\right]\psi = \left(\frac{\partial}{\partial x}\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x^2}\frac{\partial}{\partial x}\right)\psi = \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^3}{\partial x^3}\right)\psi = 0$

(iii) $\left[\frac{\partial}{\partial x}, F(x)\right]\psi = \frac{\partial}{\partial x}(F\psi) - F\frac{\partial}{\partial x}\psi = \frac{\partial F}{\partial x}\psi + F\frac{\partial\psi}{\partial x} - F\frac{\partial\psi}{\partial x} = \frac{\partial F}{\partial x}\psi$

Thus, $\left[\frac{\partial}{\partial x}, F(x)\right] = \frac{\partial F}{\partial x}$

**Example 15:** Find the equivalence of the following operators:

(i) $\left(\frac{d}{dx} + x\right)\left(\frac{d}{dx} + x\right)$ (ii) $\left(\frac{d}{dx} + x\right)\left(\frac{d}{dx} - x\right)$

**Soln.** (i) $\left(\frac{d}{dx} + x\right)\left(\frac{d}{dx} + x\right)\psi(x) = \left(\frac{d}{dx} + x\right)\left(\frac{d\psi}{dx} + x\psi\right)$

$$= \frac{d^2\psi}{dx^2} + x\frac{d\psi}{dx} + \psi + x\frac{d\psi}{dx} + x^2\psi = \left(\frac{d^2}{dx^2} + 2x\frac{d}{dx} + x^2 + 1\right)\psi$$

Therefore, $\left(\frac{d}{dx} + x\right)\left(\frac{d}{dx} + x\right) = \frac{d^2}{dx^2} + 2x\frac{d}{dx} + x^2 + 1$

(ii) Similarly, $\left(\frac{d}{dx} + x\right)\left(\frac{d}{dx} - x\right) = \frac{d^2}{dx^2} - x^2 - 1$

**Example 16.** By what factors do the operators $(x^2p_x^2+p_x^2x^2)$ and $1/2(xp_x+p_xx)^2$ differ?

**oln.** $(x^2p_x^2+p_x^2x^2)f=-\hbar^2\left[x^2\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2(x^2f)}{\partial x^2}\right]$

$$=-\hbar^2x^2\frac{\partial^2 f}{\partial x^2}-\hbar^2\frac{\partial}{\partial x}\frac{\partial(x^2f)}{\partial x}\quad=-\hbar^2x^2\frac{\partial^2 f}{\partial x^2}-\hbar^2\frac{\partial}{\partial x}\left(2xf+x^2\frac{\partial f}{\partial x}\right)$$

$$=-\hbar^2\left(x^2\frac{\partial^2 f}{\partial x^2}+2f+2x\frac{\partial f}{\partial x}+x^2\frac{\partial^2 f}{\partial x^2}+2x\frac{\partial f}{\partial x}\right)\quad=-\hbar^2\left(2x^2\frac{\partial^2}{\partial x^2}+4x\frac{\partial}{\partial x}+2\right)f$$

$$\frac{1}{2}(xp_x+p_xx)^2f=-\frac{i\hbar}{2}(xp_x+p_xx)\left[x\frac{\partial f}{\partial x}+\frac{\partial(xf)}{\partial x}\right]$$

$$=-\frac{i\hbar}{2}(xp_x+p_xx)\left(2x\frac{\partial f}{\partial x}+f\right)$$

$$=-\frac{\hbar^2}{2}\left[x\frac{\partial}{\partial x}\left(2x\frac{\partial f}{\partial x}\right)+x\frac{\partial f}{\partial x}+\frac{\partial}{dx}\left(2x^2\frac{\partial f}{\partial x}\right)+\frac{\partial(xf)}{\partial x}\right]$$

$$=-\frac{\hbar^2}{2}\left(2x^2\frac{\partial^2 f}{\partial x^2}+2x\frac{\partial f}{\partial x}+x\frac{\partial f}{\partial x}+2x^2\frac{\partial^2 f}{\partial x^2}+4x\frac{\partial f}{\partial x}+x\frac{\partial f}{\partial x}+f\right)$$

$$=-\frac{\hbar^2}{2}\left(8x\frac{\partial f}{\partial x}+2x^2\frac{\partial^2 f}{\partial x^2}+f\right)=-\hbar^2\left(2x^2\frac{\partial^2}{\partial x^2}+4x\frac{\partial}{\partial x}+\frac{1}{2}\right)f$$

The two operators differ by a term $-(3/2)\hbar^2$

**Example 17.** Find the eigenvalues and eigenfunctions of the operator $d/dx$.

**oln.** Eigenvalue equation: $\frac{d}{dx}\psi(x)=k\psi(x)$ (where $k$ is the eigenvalue and $\psi(x)$ is the eigenfunction)

$$\Rightarrow\frac{d\psi}{\psi}=kdx\quad\Rightarrow\psi=Ce^{kx}$$

**Case 1:** $k$ is a real positive quantity, $\psi$ is not an acceptable function since it tends to $\infty$ or $-\infty$ as $x\to\infty$ or $-\infty$.

**Case 2:** $k$ is purely imaginary (say $ia$), then $\psi=Ce^{iax}$ which will be finite for all values of x.

Hence, $y=ce^{kx}$ is the eigenfunction of the operator $d/dx$ with eigenvalues $k=ia$, where $a$ is real.

## Expectation value of dynamic variables:

It is defined as the average of the result of a large number of independent measurements of a physical observable on the same system.

$$\langle\hat{A}\rangle=\frac{\int_{-\infty}^{\infty}\psi^*\hat{A}\psi\,dx}{\int_{-\infty}^{\infty}\psi^*\psi\,dx}=\frac{\langle\psi|\hat{A}|\psi\rangle}{\langle\psi|\psi\rangle}$$

**Note:**

(1) If the state of the particle is an eigenfunction of the operator $\hat{A}$, then the expectation value of the physical observable corresponding to $\hat{A}$ will be equal to the eigenvalue of $\hat{A}$ corresponding to the state of the particle.

(2) The state of the particle is given as:

$$|\psi\rangle = C_1|\phi_1\rangle + C_2|\phi_2\rangle + C_3|\phi_3\rangle + \ldots\ldots\ldots\ldots\ldots = \sum_n C_n|\phi_n\rangle$$

where $|\phi_1\rangle, |\phi_2\rangle$.......... are the eigenfunction of the operator $\hat{A}$, then expectation value of the the physical observable corresponding to $\hat{A}$ will be

$$\langle \hat{A}\rangle = \sum_n |C_n|^2 \lambda_n$$

where $\lambda_n$ are the eigenvalues of operator $\hat{A}$ corresponding to $|\phi_n\rangle$.

**Example 18:** A particle of mass 'm' is confined in a 1-D box from $x=-2L$ to $x=2L$. The wave function of the particle in this state is $\psi(x) = \psi_0 \cos\left(\frac{\pi x}{4L}\right)$

(i) Find the normalization factor.

(ii) The expectation value of P² in this state.

**Soln.** (i) $\int\limits_{-\infty}^{\infty} \psi^*\psi\, dx = 1 \quad \Rightarrow |\psi_0|^2 \int\limits_{-2L}^{2L} \cos^2\left(\frac{\pi x}{4L}\right)dx = 1 \Rightarrow |\psi_0|^2 \int\limits_{-2L}^{2L} \frac{1}{2}\left(1+\cos\frac{2\pi x}{4L}\right)dx$

$$\Rightarrow \frac{|\psi_0|^2}{2}\left[x - \sin\left(\frac{2\pi x}{4L}\right)\times\frac{4L}{2\pi}\right]_{-2L}^{+2L} = 1 \Rightarrow \frac{|\psi_0|^2}{2}\left[(2L+2L)\right] = 1 \Rightarrow |\psi_0| = \frac{1}{\sqrt{2L}}$$

$$\langle P^2\rangle = \int_{-2L}^{2L}\psi^*\left[-\hbar^2\frac{d^2}{dx^2}\right]\psi\, dx = \frac{\hbar^2}{2L}\int_{-2L}^{2L}\cos^2\left(\frac{\pi x}{4L}\right)\left(\frac{\pi}{4L}\right)^2 dx$$

$$= \frac{\hbar^2}{2L}.\frac{\pi^2}{16L^2}\frac{1}{2}\int_{-2L}^{+2L}\left(1+\cos\frac{\pi x}{2L}\right)dx = \frac{\pi^2\hbar^2}{32L^3}\int_0^{2L}\left(1+\cos\frac{\pi x}{2L}\right)dx = \frac{\pi^2\hbar^2}{16L^2}$$

**Example 19:** Find the expected value of position and momentum of a particle whose wave function is $\psi(x) = e^{-x^2/a^2+ikx}$ in all space

**Soln.** Expected position of the particle $\langle x\rangle = \dfrac{\int_{-\infty}^{\infty}\psi^* x\psi\, dx}{\int_{-\infty}^{\infty}\psi^*\psi\, dx} = \dfrac{\int_{-\infty}^{\infty} e^{-2x^2/a^2}.x dx}{\int_{-\infty}^{\infty} e^{-2x^2/a^2}dx} = 0$

Expected momentum of the particle

$$\langle p\rangle = \frac{\int_{-\infty}^{\infty} e^{-x^2/a^2-ikx}\left(-i\hbar\frac{\partial}{\partial x}\right)e^{-x^2/a^2+ikx}dx}{\int_{-\infty}^{\infty} e^{-2x^2/a^2}dx} = -i\hbar\frac{\int_{-\infty}^{\infty} e^{-2x^2/a^2}\left(-\frac{2x}{a^2}+ik\right)dx}{e^{-2x^2/a^2}dx} = \hbar k\frac{\int\limits_{-\infty}^{\infty} e^{-2x^2/a^2}dx}{\int\limits_{-\infty}^{\infty} e^{-2x^2/a^2}dx} = \hbar k$$

**Example 20.** The wavefunction of a particle in a state is $\psi = N\exp(-x^2/2\alpha)$, where $N = (1/\pi\alpha)^{1/4}$. Evaluate $(\Delta x)(\Delta p)$.

**oln.** Since $\psi$ is symmetrical about $x = 0, \langle x \rangle = 0$

$$\langle x^2 \rangle = N^2 \int_{-\infty}^{\infty} x^2 \exp\left(\frac{-x^2}{\alpha}\right) dx = \frac{\alpha}{2}$$

Since, $\psi$ is a real wave function, $\langle p \rangle = 0$

$$\langle p^2 \rangle = (-i\hbar)^2 N^2 \int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{2\alpha}\right) \frac{d^2}{dx^2} \exp\left(\frac{-x^2}{2\alpha}\right) dx$$

$$= \frac{\hbar^2 N^2}{\alpha} \int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{\alpha}\right) dx - \frac{\hbar^2 N^2}{\alpha^2} \int_{-\infty}^{\infty} x^2 \exp\left(\frac{-x^2}{\alpha}\right) dx$$

$$= \frac{\hbar^2}{\alpha} - \frac{\hbar^2}{2\alpha} = \frac{\hbar^2}{2\alpha}$$

Therefore, $\Delta x \Delta p = \left[\sqrt{\langle x^2 \rangle - \langle x \rangle^2}\right]\left[\sqrt{\langle p^2 \rangle - \langle p \rangle^2}\right] = \sqrt{\frac{\alpha}{2}}\sqrt{\frac{\hbar^2}{2\alpha}} = \frac{\hbar}{2}$

**Example 21.** The Hamiltonian operator of a system is $H = -(d^2/dx^2) + x^2$. Show that $Nx\exp(-x^2/2)$ is an eigenfunction of H and determine the eigenvalue. Also evaluate N by normalization of the function.

**Soln.** $\psi = Nx\exp(-x^2/2), N$ being a constant

$$H\psi = \left(-\frac{d^2}{dx^2} + x^2\right) Nx\exp\left(-\frac{x^2}{2}\right) = Nx^3 \exp\left(-\frac{x^2}{2}\right) \frac{d}{dx}\left[\exp\left(-\frac{x^2}{2}\right) - x^2 \exp\left(-\frac{x^2}{2}\right)\right]$$

$$= 3Nx\exp\left(-\frac{x^2}{2}\right) = 3\psi$$

Thus, the eigenvalue of $H$ is 3. The normalization condition gives

$$N^2 \int_{-\infty}^{\infty} x^2 e^{-x^2} dx = 1 \Rightarrow N^2 \frac{\sqrt{\pi}}{2} = 1 \Rightarrow N = \left(\frac{2}{\sqrt{\pi}}\right)^{1/2}$$

The normalized function $\psi = \left(\frac{2}{\sqrt{\pi}}\right)^{1/2} x\exp\left(-\frac{x^2}{2}\right)$

**Example 22.** Consider the wave function $\psi(x) = A\exp\left(-\frac{x^2}{a^2}\right)\exp(ikx)$, where $A$ is a real constant.

(i) Find the value of A, (ii) calculate $\langle p \rangle$ for this wave function.

**Soln.** (i) Normalization condition: $A^2 \int_{-\infty}^{\infty} \exp\left(-\frac{2x^2}{a^2}\right) dx = 1$

$$\Rightarrow A^2\left(\frac{\pi}{2/a^2}\right)^{1/2}=1 \quad \Rightarrow A=\sqrt{\frac{2}{\pi}}\,a$$

(ii) $$\langle p\rangle=\int\psi^*\left(-i\hbar\frac{d}{dx}\right)\psi dx \;=(-i\hbar)A^2\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{a^2}\right)e^{-ikx}\left(-\frac{2x}{a^2}+ik\right)\exp\left(-\frac{x^2}{a^2}\right)e^{-ikx}dx$$

$$=(-i\hbar)\left(-\frac{2}{a^2}\right)\int_{-\infty}^{\infty}\exp\left(\frac{-2x^2}{a^2}\right)xdx+(-i\hbar)(ik)A^2\int_{-\infty}^{\infty}\exp\left(\frac{-2x^2}{a^2}\right)dx$$

In the first term, the integrand is odd and the integral is from $-\infty$ to $\infty$. Hence the integral vanishes.

Therefore, $\langle p\rangle=\hbar k$, since $A^2\int_{-\infty}^{\infty}\exp\left(\frac{-2x^2}{a^2}\right)dx=1$

**Example 23.** The normalized wave function of a particle is $\psi(x)=A\exp(iax-ibt)$, where $A$, $a$ and $b$ are constants. Evaluate the uncertainty in its momentum.

**Soln.** $$\langle p\rangle=-i\hbar\int\psi^*\frac{d}{dx}\psi dx=\hbar a\int\psi^*\psi dx=\hbar a$$

$$\langle p^2\rangle=-\hbar^2\int\psi^*\frac{d^2}{dx^2}\psi dx=-\hbar^2(ia)^2\int\psi^*\psi\, dx=\hbar^2a^2$$

Uncertainty in mometum $(\Delta p)=\sqrt{\langle p^2\rangle-\langle p\rangle^2}=\sqrt{\hbar^2a^2-\hbar^2a^2}=0$

**Example 24:** A particle of mass 'm' is subjected to the potential energy $V(x)=\frac{1}{2}kx^2$. At a particular time, it has the wave function $\psi(x)=Axe^{-x^2/a^2}$. If the particle has a definite total mechanical energy, find the value of 'a'.

**Soln.** The operator corresponding to the total mechanical energy is $H=-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)$. If the particle has definite vlaue of the total mechanical energy, its wave function should be an eigenfunction of H, that is,

$H\psi(x)=\lambda\psi(x)$. where $\lambda$ is independent of $x$.

$$H\psi(x)=\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{1}{2}kx^2\right]\left[Axe^{-x^2/a^2}\right]$$

$$=-\frac{A\hbar^2}{2m}\frac{d^2}{dx^2}\left(xe^{-x^2/a^2}\right)+\frac{1}{2}kAx^3e^{-x^2/a^2}$$

$$=-\frac{A\hbar^2}{2m}\frac{d}{dx}\left[\left(1-\frac{2x^2}{a^2}\right)e^{-x^2/a^2}\right]+\frac{1}{2}kAx^3e^{-x^2/a^2}$$

$$=-\frac{A\hbar^2}{2m}\left[\left(1-\frac{2x^2}{a^2}\right)\left(-\frac{2x}{a^2}\right)-\frac{4x}{a^2}\right]e^{-x^2/a^2}+\frac{1}{2}kAx^3e^{-x^2/a^2}$$

$$= -\frac{A\hbar^2}{2m}\left[-\frac{6x}{a^2}+\frac{4x^3}{a^4}\right]e^{-x^2/a^2}+\frac{1}{2}kAx^3e^{-x^2/a^2}$$

$$=\frac{3A\hbar^2}{ma^2}xe^{-x^2/a^2}-\left(\frac{2A\hbar^2}{ma^4}-\frac{1}{2}kA\right)x^3e^{-x^2/a^2}$$

If $H\psi(x)$ has to have the same functional form as $\psi(x)$, one should not have the $x^3e^{-x^2/a^2}$ term. So,

$$\frac{2A\hbar^2}{ma^4}-\frac{1}{2}kA=0 \quad \Rightarrow a=\left[\frac{4\hbar^2}{mk}\right]^{1/4}$$

## 4.4 Schrodinger Equation :

Consider a particle of mass 'm' is moving along $x$ axis, under the potential field $V(x)$ which is independent of time. The total energy of the system can be written as

$$E=E_k+E_p=\frac{p_x^2}{2m}+V(x)$$

Now, the 1-D time-dependent Schrodinger equation for the system can be written as

$$\hat{H}\psi=\hat{E}\psi \qquad \Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}+V(x)\psi=i\hbar\frac{\partial\psi}{\partial t}$$

Solution of 1-D time-dependent Schrodinger equation:

Let, $\psi(x,t)=X(x)T(t)$

Replecing this in the 1-D time-dependent Schrodinger equation

$$\Rightarrow -\frac{\hbar^2}{2m}T\frac{\partial^2X}{\partial x^2}+VXT=i\hbar X\frac{\partial T}{\partial t}$$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{1}{X}\frac{d^2X}{dx^2}+V=\frac{i\hbar}{T}\frac{dT}{dt}=E\ (\text{constant})$$

**Time dependent part:**

$$i\hbar\frac{1}{T}\frac{\partial T}{\partial t}=E \quad \Rightarrow\frac{\partial T}{T}=\frac{E}{i\hbar}\partial t=-\frac{i}{\hbar}Edt$$

$$\Rightarrow T(t)=Ce^{-i\frac{E}{\hbar}t}$$

Therefore, the time dependent part of the solution of the 1-D Schrodinger equation is independent of $V(x)$.

**Time independent part:**

$$-\frac{\hbar^2}{2m}\frac{1}{X}\frac{\partial^2X}{\partial x^2}+V=E \Rightarrow -\frac{\hbar^2}{2m}\frac{d^2X}{dx^2}+VX=EX$$

$$\Rightarrow\frac{d^2X}{dx^2}+\frac{2m}{\hbar^2}\left[E-V(x)\right]X=0$$

Therefore, the time independent part of the solution of the 1-D schrodinger equation depends on the nature of $V(x)$.

So, the solution of the 1-D time dependent Schrodinger equation can be written as

$$\psi(x,t)=X(x)T(t)=X(x)e^{-iEt/\hbar}$$

Now, if $\psi_1(x,t), \psi_2(x,t)......\psi_n(x,t)$ are solutions of 1-D time dependent Schrodinger equation corresponding to energies $E_1, E_2, E_3,.....E_n$, then the general solution is

$$\psi(x,t) = \psi_1(x,t) + \psi_2(x,t) + ........$$

$$= C_1X_1(x)e^{-iE_1t/\hbar} + C_2X_2e^{-iE_2t/\hbar} + C_3X_3e^{-iE_3t/\hbar} + ....... = \sum_n C_nX_n(x)e^{-iE_nt/\hbar}$$

**Example 25:** The wave function of a spinless particle of mass 'm' is 1-D potential is $\psi(x) = A\exp(-\alpha^2x^2)$, corresponding to eigenvalue $E_0 = \frac{\hbar^2\alpha^2}{m}$. The potential $V(x)$ is

(a) $2E_0(1-\alpha^2x^2)$ (b) $2E_0(1+\alpha^2x^2)$ (c) $2E_0\alpha^2x^2$ (d) $2E_0\alpha^2x^2(1+2\alpha^2x^2)$

**Soln.** Given: $\psi(x) = A\exp(-\alpha^2x^2) \Rightarrow \frac{\partial\psi}{\partial x} = A\exp(-\alpha^2x^2)(-2\alpha^2x)$

$$\Rightarrow \frac{\partial^2\psi}{\partial x^2} = A\exp(-\alpha^2x^2)(4\alpha^4x^2) + A\exp(-\alpha^2x^2)(-2\alpha^2)$$

Replacing all these in the 1-D time independent Schrodinger equation i.e

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = E\psi$$

we get

$$\Rightarrow -\frac{\hbar^2}{2m}\left[A\exp(-\alpha^2x^2)(4\alpha^4x^2)\right] + A\exp(-\alpha^2x^2)V(x) + A\exp(-\alpha^2x^2)(-2\alpha^2) = E.A\exp(-\alpha^2x^2)$$

$$\Rightarrow -\frac{\hbar^2}{2m}\left[4\alpha^4x^2 - 2\alpha^2\right] + V(x) = E \quad \Rightarrow V(x) - \frac{\hbar^2}{2m}4\alpha^4x^2 + \frac{\hbar^2\alpha^2}{m} = \frac{\hbar^2\alpha^2}{m}$$

Therefore, $V(x) = \frac{2\alpha^4\hbar^2x^2}{m} = 2E_0\alpha^2x^2$

**Correct option is (c).**

**Example 26:** The wave function of a particle moving in a 1-D potential $V(x)$ is $\psi(x) = \exp(-iax+b)$. The potential $V(x)$ will be the form:

(a) $V(x) \propto x^2$ (b) $V(x) \propto x$ (c) $V(x) = 0$ (d) $V(x) \propto e^{-ax}$

**Soln.** $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = E\psi \Rightarrow -\frac{\hbar^2}{2m}a^2 + V(x) = E \qquad \Rightarrow V(x) = 0$

**Correct option is (c).**

**Example 27:** The ground state (a part from normalization) of a particle moving in a 1-D potential given by, $\psi(x) = \exp\left(-\frac{x^2}{2}\right)\cosh(\sqrt{2}x)$. The potential $V(x)$ is (taking $m = 1$, $\hbar = 1$)

(a) $\frac{x^2}{2}$ (b) $\frac{x^2}{2} - \sqrt{2}\tanh(\sqrt{2}x)$ (c) $\frac{x^2}{2} - \sqrt{2}\tan(\sqrt{2}x)$ (d) $\frac{x^2}{2} - \coth(\sqrt{2}x)$

**oln.** Given: $\psi = \exp\left(-\frac{x^2}{2}\right)\cosh\left(\sqrt{2}x\right)$

$$\psi' = \exp\left(-\frac{x^2}{2}\right)(-x)\cosh\left(\sqrt{2}x\right) + \exp\left(-\frac{x^2}{2}\right)\sqrt{2}\sinh\sqrt{2}x$$

$$\psi'' = \exp\left(-\frac{x^2}{2}\right)\left(x^2\right)\cosh\left(\sqrt{2}x\right) - \exp\left(-\frac{x^2}{2}\right)\cosh\sqrt{2}x - \sqrt{2}x\exp\left(-\frac{x^2}{2}\right)\sinh\left(\sqrt{2}x\right)$$

$$+\exp\left(-\frac{x^2}{2}\right)(-x)\sqrt{2}\sinh\sqrt{2}x + \exp\left(-\frac{x^2}{2}\right)2\cosh\sqrt{2}x$$

Replacing all these in the 1-D time independent Schrodinger equation, i.e

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = E\psi$$

we get

$$\Rightarrow \; -\frac{\hbar^2}{2m}\left[x^2\cosh\left(\sqrt{2x}\right) - \cosh\sqrt{2}x - \sqrt{2}x\right]\sinh\sqrt{2}x - x\sqrt{2}\sinh\sqrt{2}x$$

$$+2\cosh\sqrt{2}x + V\cosh\left(\sqrt{2}x\right) = E\cosh\left(\sqrt{2x}\right)$$

$$\Rightarrow \; -\frac{\hbar^2}{2m}\left[x^2 - 1 - 2\sqrt{2}\tanh\left(\sqrt{2}x\right) + 2\right] + V = E$$

Since, energy of the particle $E$ is independent of $x$, then

$$\Rightarrow \; V(x) - \frac{\hbar^2}{2m}\left(x^2 - 2\sqrt{2}\tanh\left(\sqrt{2}x\right)\right) = 0 \;\; \Rightarrow \; V(x) = \frac{\hbar^2}{2m}\left(x^2 - 2\sqrt{2}\tanh\left(\sqrt{2}x\right)\right)$$

Taking $m = 1, \hbar = 1$ we get, $\Rightarrow V(x) = \frac{x^2}{2} - \sqrt{2}\tanh\left(\sqrt{2}x\right)$

**Superposition of States:**

If $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle ......|\phi_n\rangle$ are orthonormal state vectors/wave functions and are representing state of quantum mechanics system. then any linear combination will also represent state of a quantum mechanics system. In other words, if $\phi_1, \phi_2, .......\phi_n$ are solutions of Schrodinger equation, then linear combination/superposition of the states i.e.

$$|\psi\rangle = C_1|\phi_1\rangle + C_2|\phi_2\rangle + C_3|\phi_3\rangle + ................ = \sum_n C_n|\phi_n\rangle$$

will also be a solution of Schrodinger equation.

Note:

(1) $|\psi\rangle$ will be normalized if $|C_1|^2 + |C_2|^2 + |C_3|^2 ................ = 1$

(2) Probability of finding a particle in $|\phi_1\rangle$ state $= |C_1|^2 = |\langle\phi_1|\psi\rangle|^2$

**Example 28:** Consider a system whose initial state at t = 0 is given in terms of 3 orthonormal state vectors

$|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle$ as

$$|\psi(x,0)\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle + A|\phi_2\rangle + \frac{1}{\sqrt{5}}|\phi_3\rangle$$ (A is a real constant)

(i) Find the value of A so that $|\psi(x,0)\rangle$ is normalized.

(ii) What are probabilities regarding the system to be in $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle$ states.

(iii) Find the state of the system at any time 't'.

**Soln.** (i) Normalization condition:

$$\langle\psi|\psi\rangle = 1 \quad \Rightarrow \sum |C_n|^2 = 1 \quad \Rightarrow \frac{1}{2} + A^2 + \frac{1}{5} = 1 \Rightarrow A^2 = \frac{1}{2} - \frac{1}{5} = \frac{3}{10} \quad \Rightarrow A = \sqrt{\frac{3}{10}}$$

Therefore, $|\psi(x,0)\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle + \sqrt{\frac{3}{10}}|\phi_1\rangle + \frac{1}{\sqrt{5}}|\phi_3\rangle$

Probabilities regarding the system to be in $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle$ states are

$$P_1 = |C_1|^2 = \frac{1}{2},\ P_2 = |C_2|^2 = \frac{3}{10},\ P_3 = |C_3|^2 = \frac{1}{5}$$

The state of the system at any time 't' will be

$$|\psi(x,t)\rangle = \frac{1}{\sqrt{2}}e^{-iE_1t/\hbar}|\phi_1\rangle + \sqrt{\frac{3}{10}}e^{-iE_2t/\hbar}|\phi_2\rangle + \frac{1}{\sqrt{5}}e^{-iE_3t/\hbar}$$

**Example 29:** The wavefunction of a particle is given by $|\psi_n\rangle = \frac{1}{\sqrt{2}}|\phi_0\rangle + i|\phi_1\rangle$ where $|\phi_0\rangle$ and $|\phi_1\rangle$ are the normalized eigenstates with energies $E_0$ and $E_1$ respectively. The hamiltonian in the state $\psi$ is

(a) $\frac{E_0}{2} + E_1$ (b) $\frac{E_0}{2} - E_1$ (c) $\frac{E_0 - 2E_1}{3}$ (d) $\frac{E_0 + 2E_1}{3}$

**Soln.** Suppose normalized wavefunction: $|\psi\rangle = C\left[\frac{1}{\sqrt{2}}|\phi_0\rangle + |\phi_1\rangle\right]$ (where C is normalization constnat)

Normalization condition: $\langle\psi|\psi\rangle = 1 \quad \Rightarrow \frac{C^3}{2} + C^2 = 1 \quad \Rightarrow C = \sqrt{\frac{2}{3}}$

Therefore, the normalized wavefunction will be

$$|\psi\rangle = \frac{1}{\sqrt{3}}|\phi_0\rangle + \sqrt{\frac{2}{3}}|\phi_2\rangle$$

Energy of the particle in the given state will be

$$\langle E\rangle = P(\phi_0)E_0 + P(\phi_1)E_1 = \frac{E_0 + 2E_1}{3}$$

## 4.5 Probability Current density:

As the time changes from t = 0, the probability of finding the particle in some region of space may increase or decrease. If the probability increases in some region, then it should decrease in some other region such that total probability of finding the particle in the entire space should be equal to one. We can assume this as a flow of probability from one region to another region, like a fluid or current. Therefore, the probability flow satisfies the equation of continuity i.e.

$$\vec{\nabla}.\vec{J} + \frac{\partial\rho}{\partial t} = 0$$

where $\rho =$ probability density $= \psi^* \psi$

and $\vec{J} =$ probability current density $= -\dfrac{i\hbar}{2m}\left[\psi^* \vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right]$

The magnitude of probabilty current density represents the flux of the particles i.e number of particles passing through per unit area per unit time and the direction of probability current density is along the direction of flow of the particles.

**Example 30.** A particle of mass 'm' is reprensented by wave function $\psi(x) = Ae^{ikx}$ ($k$ = wave number, $A$ = constant). The magnitude of probability current density is

(a) $|A|^2 \dfrac{\hbar k}{m}$ (b) $|A|^2 \dfrac{\hbar k}{2m}$ (c) $|A|^2 \dfrac{(\hbar k)^2}{m}$ (d) $|A|^2 \dfrac{(\hbar k)^2}{2m}$

**Soln.** Probability current density =

$$\vec{J} = -\frac{i\hbar}{2m}\left[\psi^* \frac{\partial \psi}{\partial x} - \psi\frac{\partial \psi^*}{\partial x}\right]\hat{x} = -\frac{i\hbar}{2m}\left[|A|^2 ik + |A|^2 ik\right]\hat{x} = |A|^2 \frac{\hbar k}{m}\hat{x}$$

**Correct answer is (a)**

**Example 31.** The wave function of particle moving in free space is $\psi = e^{ikx} + 2e^{-ikx}$

(A) The probability current density for the real part of the wave function

(a) 1 (b) $\dfrac{\hbar k}{m}$ (c) $\dfrac{\hbar k}{2m}$ (d) 0

(B) The energy of the particle is

(a) $\dfrac{5\hbar^2 k^2}{2m}$ (b) $\dfrac{3\hbar^2 k^2}{2m}$ (c) $\dfrac{\hbar^2 k^2}{2m}$ (d) $\dfrac{\hbar^2 k^2}{m}$

**Soln.** **(A)** The real part of the wave function $\psi = e^{ikx} + 2e^{-ikx}$ is $3\cos kx$

Therefore, the corresponding probability density will be zero.

**Correct answer is (d)**

**(B)** $$\hat{H}\psi = \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\psi = -\frac{\hbar^2}{2m}\left[\frac{d}{dx}\left\{ik\, e^{+ikx} - 2ik\, e^{-ikx}\right\}\right]$$

$$= -\frac{\hbar^2}{2m}\left[-k^2 e^{ikx} - 2k^2 e^{-ikx}\right] = \frac{\hbar^2 k^2}{2m}\psi$$

**Correct answer is (c)**

**Example 32.** Calculate the probability current density for a Gaussian wave pocket

$$\psi(x) = A\exp\left[-\frac{x^2}{a^2} + \frac{ip_0 x}{\hbar}\right]$$

**Soln.** $$\vec{J} = -\frac{i\hbar}{2m}\left[\psi^* \frac{\partial \psi}{\partial x} - \psi\frac{\partial \psi^*}{\partial x}\right]\hat{x}$$

$$= -\frac{i\hbar}{2m}\left[A^* \exp\left(-\frac{x^2}{a^2} - i\frac{p_0 x}{\hbar}\right)\left(-\frac{2x}{a^2} + \frac{ip_0}{\hbar}\right)A\exp\left(-\frac{x^2}{a^2} + i\frac{p_0 x}{\hbar}\right)\right.$$

$$\left. - A\exp\left(-\frac{x^2}{a^2} + i\frac{p_0 x}{\hbar}\right)\left(-\frac{2x}{a^2} - \frac{ip_0}{\hbar}\right)A^* \exp\left(-\frac{x^2}{a^2} - i\frac{p_0 x}{\hbar}\right)\right]\hat{x}$$

$$=-\frac{i\hbar}{2m}\left[|A|^2\exp\left(-\frac{2x^2}{a^2}\right)\left\{-\frac{2x}{a^2}+\frac{ip_0}{\hbar}+\frac{2x}{a^2}+\frac{ip_0}{\hbar}\right\}\right]\hat{x}$$

$$=\left(-\frac{i\hbar}{2m}\right)\left(\frac{2ip_0}{\hbar}\right)|A|^2\exp\left(-\frac{2x^2}{a^2}\right)\hat{x}=\frac{p_0}{m}|A|^2\exp\left(-\frac{2x^2}{a^2}\right)\hat{x}$$

**Example 33.** Calculate the probability current density for a 3-D wave function $\psi(r)=\frac{A}{r}e^{ikr}$

**Soln.** $$\vec{J}=-\frac{i\hbar}{2m}\left[\psi^*\frac{\partial\psi}{\partial r}-\psi\frac{\partial\psi^*}{\partial r}\right]\hat{r}$$

$$\vec{J}=-\frac{i\hbar}{2m}\left[\frac{A}{r}e^{-ikr}A\left\{\frac{ik}{r}e^{ikr}-\frac{e^{ikr}}{r^2}\right\}-\frac{A}{r}e^{ikr}.A\left\{-\frac{ik}{r}e^{-ikr}-\frac{1}{r^2}e^{-ikr}\right\}\right]\hat{r}$$

$$=-\frac{i\hbar}{2m}\left[\frac{A^2}{r^2}\left(ik-\frac{1}{r}\right)+\frac{A^2}{r^2}\left(ik+\frac{1}{r}\right)\right]=-\frac{i\hbar}{2m}2\frac{A^2}{r^2}ik=\frac{\hbar k}{m}\frac{|A|^2}{r^2}$$

**Example 34.** Show that for any normalized wave packet in one dimesion $\int_{-\infty}^{\infty}J(x)dx=\langle p_x\rangle/m$.

**Soln.** $$\int_{-\infty}^{\infty}J(x)dx=\frac{\hbar}{2im}\left[\int_{-\infty}^{\infty}\psi^*\frac{\partial}{\partial x}\psi dx-\int_{-\infty}^{\infty}\psi\frac{\partial}{\partial x}\psi^* dx\right]$$

$$=\frac{\hbar}{2im}\left[\frac{1}{-i\hbar}\int_{-\infty}^{\infty}\psi^* dx\left(-i\hbar\frac{d}{dx}\right)\psi(x)-c.c.\right]$$

where c.c. is the complex conjugate of the first term. Now, $\langle p_x\rangle$ must be real. So, the first term in the square bracket is $\frac{1}{-i\hbar}\langle p_x\rangle$ and hence its complex conjugate is $\frac{1}{i\hbar}\langle p_x\rangle$.

Thus, $$\int_{-\infty}^{\infty}J(x)dx=\frac{\hbar}{2im}\left[\frac{2}{-i\hbar}\langle p_x\rangle\right]=\frac{\langle p_x\rangle}{m}$$

**Example 35.** The solution of the Schrodinger equation for a free particle of mass $m$ in one dimension is $\psi(x,t)$. At $t=0$, $\psi(x,0)=A\exp\left(\frac{-x^2}{a^2}\right)$. Find the probability amplitude in momentum space at $t=0$ and at time $t$.

**Soln.** $$\Phi(p,0)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi(x,0)\exp\left(-\frac{ipx}{\hbar}\right)dx$$

$$=\frac{A}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\exp\left(-\frac{x^2}{a^2}-\frac{ipx}{\hbar}\right)dx\quad=\frac{Aa}{\sqrt{2\hbar}}\exp\left(-\frac{p^2a^2}{4\hbar^2}\right)$$

The Schrodinger equation for the momentum space equation is

$$i\hbar\frac{\partial\Phi}{\partial t}=\frac{p^2}{2m}\Phi\Rightarrow\frac{d\Phi}{\Phi}=\left(-\frac{ip^2}{2m\hbar}\right)dt\Rightarrow\Phi(p,t)=C\exp\left(-\frac{ip^2t}{2m\hbar}\right)$$

At $t=0, \Phi(p,0)=C$. Hence,

$$\Phi(p,t)=\frac{Aa}{\sqrt{2\hbar}}\exp\left(-\frac{p^2a^2}{4\hbar^2}-\frac{ip^2t}{2m\hbar}\right)$$

**Example 46.** If $u_1$ and $u_2$ are eigenstates corresponding to energy eigenvalues $E_1$ and $E_2 (E_1 \neq E_2)$ of a Hamiltonian, then in the state space

(a) $u_1$ and $u_2$ are parallel **[BHU 2010, 2011]**

(b) $u_1$ and $u_2$ are orthogonal

(c) $u_1$ and $u_2$ are at an angle $\theta$ such that $\cos\theta=\frac{E_1}{E_2}$

(d) $u_1$ and $u_2$ are degenerate states

**Soln.** The eigenstates corresponding to different eigenvalues $(E_1 \neq E_2)$ of a Hermitian operator, are orthogonal.

**Correct option is (b)**

**Example 47.** If an operator satisfies the equation $\hat{O}\psi = a\psi$, then measurement of $\hat{O}$ will give

(a) $a$ in the state $\psi$ (b) the value $=\int\psi^* o\psi\, d^3x$ in the state $\psi$

(c) any value of several values greater than $a$ (d) zero **[BHU 2010]**

**Soln.** The measurement of $\hat{O}$ will give the eigenstate $\psi$ with the corresponding eigenvalue $a$.

**Correct option is (a)**

**Example 48.** The value of the commutation relation $[\hat{p}, (\hat{x}^2+\hat{p}^2)]$ is

(a) $-i\hbar$ (b) $-i\hbar\hat{x}$ (c) $-i\hbar(\hat{x}+\hat{p})$ (d) $-2i\hbar\hat{x}$ **[BHU 2010]**

**Soln.** $\left[\hat{p}, (\hat{x}^2+\hat{p}^2)\right]=\left[\hat{p}, \hat{x}^2\right]+\left[\hat{p}, \hat{p}^2\right]$

$$=[p, \hat{x}]\hat{x}+\hat{x}[\hat{p}, \hat{x}] \qquad \left[\because \left[\hat{p}, \hat{p}^2\right]=0\right]$$

$$=-i\hbar\hat{x}+\hat{x}(-i\hbar) \qquad \left[\because [\hat{x}, \hat{p}]=i\hbar\right]$$

$$=-2i\hbar\hat{x}$$

**Correct option is (d)**

**Example 49.** If $\psi_1, \psi_2$ are eigenvectors of an operator $\hat{A}$ with eigenvalues as ½ and ½ respectively, then $a\psi_1+b\psi_2$ (where $a, b$ are complex numbers) is an **[BHU 2010]**

(a) eigenvector of $\hat{A}$ with eigenvalue $\frac{a-b}{2}$ (b) eigenvector of $\hat{A}$ with eigenvalue zero

(c) eigenvector of $\hat{A}^2$ with eigenvalue $\frac{a+b}{4}$ (d) eigenvector of $\hat{A}^2$ with eigenvalue $\frac{1}{4}$

**Soln.** Given : $\hat{A}\psi_1 = \frac{1}{2}\psi_1$ and $\hat{A}\psi_2 = -\frac{1}{2}\psi_2$

Now, $\hat{A}^2(a\psi_1 + b\psi_2) = a\hat{A}^2\psi_1 + b\hat{A}^2\psi_2$

$$= \frac{a}{4}\psi_1 + \frac{b}{4}\psi_2$$

$$= \frac{1}{4}(a\psi_1 + b\psi_2)$$

$\Rightarrow$ $a\psi_1 + b\psi_2$ is an eigenvector of $\hat{A}^2$ with eigenvalue 1/4.

**Correct option is (d)**

**Example 50.** If wave function in coordinate space is $\psi(x) = \delta(x)$, then momentum space wave function is

(a) $\frac{1}{\sqrt{2\pi}}$ (b) $\delta(p)$ (c) $\frac{1}{\sqrt{2\pi}}\delta(p)$ (d) zero **[BHU 2010]**

**Soln.** Given : $\psi(x) = \delta(x)$. The momentum space wave function is

$$\psi(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x)\,e^{\frac{ipx}{\hbar}}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)\,e^{\frac{ipx}{\hbar}}\,dx = \frac{1}{\sqrt{2\pi}}.$$

**Correct option is (a)**

**Example 51.** The expectation value of the momentum $p$ in one-dimensional motion is **[BHU 2011]**

(a) $\int\psi^*\left(-i\hbar\frac{\partial\psi}{\partial t}\right)dx$ (b) $\int\psi^*\left(-\hbar^2\frac{\partial^2\psi}{\partial t^2}\right)dx$ (c) $\int\psi^*\left(-\hbar^2\frac{\partial^2\psi}{\partial x^2}\right)dx$ (d) $\int\psi^*\left(-i\hbar\frac{\partial\psi}{\partial x}\right)dx$

**Soln.** $\langle p_x\rangle = \int_{-\infty}^{\infty}\psi^*\hat{p}_x\psi\,dx = \int_{-\infty}^{\infty}\psi^*(-i\hbar)\frac{\partial\psi}{\partial x}dx$

**Correct option is (d)**

**Example 52.** If $\psi(r,t)$ represents the wave function of a particle, then to which function it would collapse just after a precise measurement of momentum? **[BHU 2011]**

(a) $\delta(\vec{r}-\vec{r}\,')$ (b) $\delta(\vec{P}-\vec{P}\,')$

(c) $\psi'(r,t)$ with smaller $\Delta p$ (d) $\psi'(r,t)$ with smaller $\Delta x$

**Soln.** For the wavefunction $\delta(\vec{P}-\vec{P}\,')$, uncertainty in momentum will be zero i.e. precise measurement of momentum is possible.

**Correct option is (b)**

**Example 53.** The wave function of a particle is given by $\psi = Ce^{-\alpha^2x^2}$ in the range $-\infty < x < \infty$, where C and $\alpha$ are constants. The probability of finding the particle in the range $0 < x < \infty$ is **[BHU 2011]**

(a) $\frac{1}{5}$ (b) $\frac{2}{4}$ (c) $\frac{1}{4}$ (d) $\frac{1}{3}$

**oln.** $$P(0<x<\infty)=\frac{\int\limits_0^\infty \psi^*\psi\,dx}{\int\limits_{-\infty}^\infty \psi^*\psi\,dx}=\frac{C^2\int\limits_0^\infty e^{-2\alpha^2x^2}\,dx}{C^2\int\limits_{-\infty}^\infty e^{-2\alpha^2x^2}\,dx}=\frac{\int\limits_0^\infty e^{-2\alpha^2x^2}\,dx}{2\int\limits_0^\infty e^{-2\alpha^2x^2}\,dx}=\frac{1}{2}$$

**Correct option is (b)**

**Example 54.** Which of the following quantum mechanical operators is Hermitian? **[BHU 2011]**

(a) $i\frac{d}{dx}$ (b) $\frac{d}{dx}$ (c) $\frac{d^2}{dx^2}$ (d) $-\frac{d}{dx}$

**oln.** $\left(\frac{d}{dx}\right)^\dagger=-\frac{d}{dx}$; $\left(-\frac{d}{dx}\right)^\dagger=\frac{d}{dx}$ i.e. these operators are skew-hermitian.

$\left(i\frac{d}{dx}\right)^\dagger=(-i)\left(-\frac{d}{dx}\right)=i\frac{d}{dx}$; $\left(\frac{d^2}{dx^2}\right)^\dagger=\left(\frac{d}{dx}\right)^\dagger\left(\frac{d}{dx}\right)^\dagger=\left(-\frac{d}{dx}\right)\left(-\frac{d}{dx}\right)=\frac{d^2}{dx^2}$ i.e these operators are hermitian.

**Correct option is (a, c).**

**Example 55.** The value of the commutation relation $\left[\hat{p},\left(\hat{x}^2+\hat{p}^2\right)\right]$ is **[BHU 2011]**

(a) $-i\hbar\hat{p}$ (b) $-i\hbar\hat{x}$ (c) $-i\hbar\left(\hat{x}+\hat{p}\right)$ (d) $-2i\hbar\hat{x}$

**oln.** $$\begin{aligned}\left[\hat{p},(\hat{x}^2+\hat{p}^2)\right]&=\left[\hat{p},\hat{x}^2\right]+\left[\hat{p},\hat{p}^2\right]\\&=[p,\hat{x}]\hat{x}+\hat{x}[\hat{p},\hat{x}] \qquad \left[\because \left[\hat{p},\hat{p}^2\right]=0\right]\\&=-i\hbar\hat{x}+\hat{x}(-i\hbar) \qquad \left[\because [\hat{x},\hat{p}]=i\hbar\right]\\&=-2i\hbar\hat{x}\end{aligned}$$

**Correct option is (d)**

**Example 56.** If the wave function for a particle in coordinate space is $\psi(x)=\delta(x)$, then that wave function in the momentum space will be **[BHU 2011]**

(a) $\psi(p)=\frac{1}{\sqrt{2\pi}}$ (b) $\psi(p)=\delta(p)$ (c) $\psi(p)=\frac{1}{\sqrt{2\pi}}\delta(p)$ (d) $\psi(p)=0$

**Soln.** Given : $\psi(x)=\delta(x)$. The momentum space wave function is

$$\psi(p)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\psi(x)\,e^{\frac{ipx}{\hbar}}\,dx=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\delta(x)\,e^{\frac{ipx}{\hbar}}\,dx=\frac{1}{\sqrt{2\pi}}.$$

**Correct option is (a)**

**Example 57.** The eigenvalues of a Hermitian operator are always **[BHU 2012]**

(a) real (b) imaginary (c) degenerate (d) linear

**Soln.** The eigenvalues of a Hermitian operator are always real.

**Correct option is (a)**

**Example 58.** The state $\psi = \frac{1}{\sqrt{6}}\psi_{-1} + \frac{1}{\sqrt{2}}\psi_1 + \frac{1}{\sqrt{3}}\psi_2$ is a linear combination of three orthonormal eigenstates of the operator $\hat{O}$ corresponding to eigenvalues −1, 1 and 2. What is the expectation value of $\hat{O}$ for this state?

(a) $\frac{2}{3}$ (b) $\sqrt{\frac{7}{6}}$ (c) 1 (d) $\frac{4}{3}$ **[BHU 2012]**

**Soln.** Given : $\psi = \frac{1}{\sqrt{6}}\psi_{-1} + \frac{1}{\sqrt{2}}\psi_1 + \frac{1}{\sqrt{3}}\psi_2$

$$\therefore \langle \hat{O} \rangle = \langle \psi | \hat{O} | \psi \rangle$$

$$= \left(\frac{1}{\sqrt{6}}\psi_{-1}^* \frac{1}{\sqrt{2}}\psi_1^* + \frac{1}{\sqrt{3}}\psi_2^*\right)\hat{O}\left(\frac{1}{\sqrt{6}}\psi_{-1} + \frac{1}{\sqrt{2}}\psi_1 + \frac{1}{\sqrt{3}}\psi_2\right)$$

$$= \left(\frac{1}{\sqrt{6}}\psi_{-1}^* + \frac{1}{\sqrt{2}}\psi_1^* + \frac{1}{\sqrt{3}}\psi_2^*\right)\left(\frac{-1}{\sqrt{6}}\psi_{-1} + \frac{1}{\sqrt{2}}\psi_1 + \frac{2}{\sqrt{3}}\psi_2\right)$$

$$= -\frac{1}{6} + \frac{1}{2} + \frac{2}{3} = -\frac{1+3+4}{6} = 1.$$

**Correct option is (c)**

**Example 59.** Which of the following functions could represent the radial wave function for an electron in an atom ? ($r$ is the distance of the electron from the nucleus, $A$ and $b$ are constants). **[BHU 2012]**

(1) $Ae^{-br}$ (2) $A\sin(br)$ (3) $\frac{A}{r}$

(a) 1 only (b) 2 only (c) 1 and 2 only (d) 1 and 3 only

**Soln.** The radial wave function should goes to zero as $r \to \infty$ and has a finite or zero value as $r \to 0$. These conditions are satisfied by only option (1). i.e., $Ae^{-br}$

**Correct option is (a)**

**Example 60.** If $\psi$ is a normalized solution of the Schrödinger equation and $Q$ is the operator corresponding to a physical observable $x$, the quantity $\psi^* Q\psi$ may be integrated in order to obtain the **[BHU 2012]**

(a) normalization constant for $\psi$ (b) spatial overlap of $Q$ with $\psi$

(c) mean value of $x$ (d) uncertainty in $x$

**Soln.** The quantity $\psi^* Q\psi$ gives the mean value of $x$.

**Correct option is (c)**

**Example 61.** Which of the following is an eigenfunction of the linear momentum operator $-i\hbar\frac{\partial}{\partial x}$ with a positive eigenvalue $\hbar k$, i.e., an eigenfunction that describes a particle that is moving in free space in the direction of positive $x$ with a precise value of linear momentum ? **[BHU 2012]**

(a) $\cos kx$ (b) $\sin kx$ (c) $e^{-ikx}$ (d) $e^{ikx}$

**Soln.** The given operator is $\hat{p} = -i\hbar\frac{d}{dx}$

For the state $\psi(x) = e^{ikx}$, we have $\hat{p}\,\psi(x) = -i\hbar\frac{d}{dx}e^{ikx} = \hbar k\,\psi(x)$

**Correct option is (d)**

**Example 62.** Which one of the following operators is Hermitian ? **[BHU 2012]**

(a) $e^{ix}$ (b) $xp_y$ (c) $\frac{\partial}{\partial x}$ (d) $\frac{i\partial}{\partial x}$

**Soln.** $\left(e^{ix}\right)^{\dagger} = e^{-ix^{\dagger}} = e^{-ix}$ i.e. not hermitian.

$\left(xp_y\right)^{\dagger} = \left(p_y\right)^{\dagger}\left(x\right)^{\dagger} = p_y x = xp_y$ i.e hermitian.

$\left(\frac{\partial}{\partial x}\right)^{\dagger} = -\frac{\partial}{\partial x}$ i.e. not hermitian.

$\left(i\frac{\partial}{\partial x}\right)^{\dagger} = \left(-i\right)\left(-\frac{\partial}{\partial x}\right) = i\frac{\partial}{\partial x}$ i.e. hermitian.

**Example 63.** According to wave mechanics, a free particle in space can possess energies of the following type **[BHU 2012]**

(a) Discrete energies (b) Continuous energies

(c) Both types of energies (d) Only one simple value of energy

**Soln.** A free particle is unbounded and hence possesses continuous energies.

**Correct option is (b)**

**Example 64.** Which of the following operators is not linear operator : **[BHU 2014]**

(a) Momentum (b) Parity (c) Time reversal (d) Angular momentum

**Soln.** Time reversal is not a linear operator.

**Correct option is (c)**

**Example 65.** Which of the following statements is incorrect about complex conjugation operator ? **[BHU 2014]**

(a) It is an Hermitian operator (b) It has eigen values $\pm 1$

(c) It is not a linear operator (d) It has eigen values $\pm 1$

**Soln.** The complex conjugation operators can take any eigenvalue.

**Correct option is (b, d)**

**Example 66.** Which of the following is a eigenstate of momentum operator ? **[BHU 2014]**

(a) $\cos kx + i\sin kx$ (b) $i\sin kx$ (c) $\cos kx$ (d) $A\cos kx + B\sin kx$

**Soln.** The momentum operator is $\hat{p} \equiv -i\hbar\frac{d}{dx}$.

$$\hat{p}\psi = -i\hbar\frac{d}{dx}(\cos kx + i\sin kx) = -\hbar k(-i\sin kx - \cos kx) = \hbar k\psi$$

**Correct option is (a)**

**Example 67.** Probabilities current density is represented by the operator : **[BHU 2014]**

(a) $\frac{i\hbar}{2m}\left(\psi\vec{\nabla}\psi^{*} - \psi^{*}\vec{\nabla}\psi\right)$ (b) $\frac{\hbar}{2im}\left(\psi\vec{\nabla}\psi^{*} - \psi^{*}\vec{\nabla}\psi\right)$

(c) $\frac{i\hbar}{2m}\left(\psi^{*}\vec{\nabla}\psi - \psi\vec{\nabla}\psi^{*}\right)$ (d) $\frac{\hbar}{2im}\left(\psi\vec{\nabla}\psi^{*} - \psi\vec{\nabla}\psi^{*}\right)$

**oln.** The probability current density is defined as

$$\vec{J}=\frac{i\hbar}{2m}\left(\psi\vec{\nabla}\psi^*-\psi^*\vec{\nabla}\psi\right)$$

**Correct option is (a)**

**Example 68.** Consider a system in a state $|\psi\rangle$ given by $|\psi\rangle=\frac{1}{\sqrt{3}}\left[|\psi_1\rangle+\sqrt{2}|\psi_2\rangle\right]\cdot|\psi\rangle$ and $|\psi_2\rangle$ are orthogonal and normalized eigen vectors of the system, with energy eigen values $E_1$ and $E_2$, respectively. If a measurement of energy is performed on the system, what is the probability of getting a value $E_2$ ?

(a) $\sqrt{\frac{2}{3}}$ (b) $\frac{2}{3}$ (c) $\sqrt{\frac{1}{3}}$ (d) $\frac{1}{3}$ **[BHU 2014]**

**ln.** Given : $|\psi\rangle=\frac{1}{\sqrt{3}}\left[|\psi_1\rangle+\sqrt{2}|\psi_2\rangle\right]$

Therefore, probability of getting a value $E_2$ is

$$P(E_2)=\frac{|\langle\psi_2|\psi\rangle|^2}{\langle\psi|\psi\rangle}=\frac{\left(\sqrt{2}/\sqrt{3}\right)^2}{\left(1/\sqrt{3}\right)^2+\left(\sqrt{2}/\sqrt{3}\right)^2}=\frac{2/3}{1}=\frac{2}{3}$$

**Correct option is (b)**

**Example 69.** If a system is invariant under parity, which of the following statements is incorrect ? **[BHU 2014]**

(a) The wave functions must have definite parity

(b) $\langle x\rangle$ is always zero

(c) $\langle p_x^2\rangle$ is always zero

(d) half of wave functions are positive under parity

**ln.** The invariance of system under parity does not say that $\langle p_x^2\rangle=0$ always.

**Correct option is (c)**

**Example 70.** The wave function of a particle at $t=0$ is given by

$$|\psi|(0)\rangle=\frac{1}{\sqrt{2}}\left[|u_1\rangle+|u_2\rangle\right],\text{ where }|u_1\rangle\text{ and }|u_2\rangle$$

are the normalized eigen states with eigen values $E_1$ and $E_2$, respectively, $(E_2>E_1)$. The shortest time after which $|\psi(t)\rangle$ will become orthonormal to $|\psi(0)\rangle$ is : **[BHU 2014]**

(a) $\frac{-\hbar\pi}{2(E_2-E_1)}$ (b) $\frac{\hbar\pi}{E_2-E_1}$ (c) $\frac{\sqrt{2}\hbar\pi}{E_2-E_1}$ (d) $\frac{2\hbar\pi}{E_2-E_1}$

**n.** **Given :** $|\psi(0)\rangle=\frac{1}{\sqrt{2}}\left[|u_1\rangle+|u_2\rangle\right]$

The eigenstate after time $t$ is

$$|\psi(t)\rangle=\frac{1}{\sqrt{2}}\left[|u_1\rangle e^{-iE_1t/\hbar}+|u_2\rangle e^{-iE_2t/\hbar}\right]$$

If $|\psi(0)\rangle$ is orthogonal to $|\psi(t)\rangle$, then $\langle\psi(0)|\psi(t)\rangle=0$

$$\Rightarrow \frac{1}{2}\left[\langle u_1| + \langle u_2|\right]\left[|u_1\rangle e^{-iE_1t/\hbar} + |u_2\rangle e^{-iE_2t/\hbar}\right] = 0$$

$$\Rightarrow e^{-iE_1t/\hbar} + e^{-iE_2t/\hbar} = 0 \quad \Rightarrow e^{-iE_1t/\hbar} = -e^{-iE_2t/\hbar}$$

$$\Rightarrow e^{i(E_2-E_1)t/\hbar} = -1 = e^{i(2n+1)\pi}\ ;\ n = 0,1,2,3$$

$$\Rightarrow \frac{(E_2-E_1)}{\hbar} = (2n+1)\pi \quad \Rightarrow t = \frac{(2n+1)\pi\hbar}{(E_2-E_1)}$$

So, for $t$ to be minimum, $n = 0$

Therefore, $t = \dfrac{\hbar\pi}{E_2 - E_1}$.

**Correct option is (b)**

**Example 71.** In a quantum system an observable is represented by an operator A. If $|\psi\rangle$ is a state of the system which is not a eigen state of A, then $r = \langle\psi | A | \psi\rangle^2 - \langle\psi | A^2 | \psi\rangle$ must be : **[BHU 2014]**

(a) equal to zero (b) greater than zero (c) less than zero (d) greater than or equal to zero

**Soln.** Given : $r = \langle\psi | A | \psi\rangle^2 - \langle\psi | A^2\psi\rangle$

The uncertainity value is

$$(\Delta A)^2 = \langle\psi | A^2\psi\rangle - \langle\psi | A | \psi\rangle^2 = -r$$

$$\Rightarrow \quad r = -(\Delta A)^2 < 0$$

**Correct option is (c)**

**Example 72.** A wave function has the form $\psi_1 = Ae^{-ax}$ for $x > 0$ for the positive value of $\alpha$. However, it has the form $\psi_2 = e^{+ax}$ for $x < 0$. The normalized form of this function is : **[BHU 2015]**

(a) $\psi_1(x) = \sqrt{\alpha}\, e^{ax}$ for $x > 0$
$\psi_2(x) = \sqrt{\alpha}\, e^{-ax}$ for $x < 0$

(b) $\psi_1(x) = \sqrt{\alpha}\, e^{+ax}$ for $x < 0$
$\psi_2(x) = \sqrt{\alpha}\, e^{-ax}$ for $x > 0$

(c) $\psi_1(x) = \dfrac{1}{\sqrt{\alpha}} e^{ax}$ for $x > 0$
$\psi_2(x) = \dfrac{1}{\sqrt{\alpha}} e^{-ax}$ for $x < 0$

(d) $\psi_1(x) = \dfrac{1}{\sqrt{\alpha}} e^{-ax}$ for $x > 0$
$\psi_2(x) = \dfrac{1}{\sqrt{\alpha}} e^{+ax}$ for $x < 0$

**Soln.** Given : $\psi(x) = \begin{cases} A e^{-\alpha x} & ; \quad x > 0 \\ A e^{+\alpha x} & ; \quad x < 0 \end{cases}$

The normalization is done by $\displaystyle\int_{-\infty}^{\infty} \psi^*(x)\,\psi(x)\,dx = 1$

$$\Rightarrow |A|^2 \int_0^{\infty} e^{-2\alpha x}\,dx + |A|^2 \int_{-\infty}^{0} e^{2\alpha x}\,dx = 1 \Rightarrow |A|^2\left(\frac{1}{2\alpha} + \frac{1}{2\alpha}\right) = 1 \Rightarrow |A| = \sqrt{\alpha}$$

$$\therefore \quad \psi(x) = \begin{cases} \sqrt{\alpha}\, e^{-\alpha x} & ; \quad x > 0 \\ \sqrt{\alpha}\, e^{\alpha x} & ; \quad x < 0 \end{cases}$$

**Correct option is (b)**

**Example 73.** If the Hamiltonian is Hermitian, two degenerate eigenstates **[BHU 2015]**

(a) would be orthogonal to each other (b) would be non-orthogonal to each other

(c) may not be orthogonal (d) may not be non-orthogonal

**Soln.** The eigenstates of a Hermitian operator are othogonal if they have different eigenvalues. So the eigenstates, if degenerate, are non-orthogonal to each other.

**Correct option is (b)**

**Example 74.** An anti-Hermitian operator corresponds to the eigenvalue that is

(a) complex (b) real (c) imaginary (d) zero **[BHU 2015]**

**Soln.** The eigenvalue corresponds to an anti-Hermitian operator is imaginary.

**Correct option is (c)**

**Example 75.** One of the following operator is hermitian : **[BHU 2016]**

(a) $\hat{x}\hat{p}_x$ (b) $\left[\hat{x}, \hat{p}_x^2\right]$ (c) $-i\frac{d^2}{dx^2}$ (d) $\frac{d^2}{dx^2}$

**Soln.** $\left(\frac{d^2}{dx^2}\right)^\dagger = \left(\frac{d}{dx}\right)^\dagger \left(\frac{d}{dx}\right)^\dagger = \left(-\frac{d}{dx}\right)\left(-\frac{d}{dx}\right) = \frac{d^2}{dx^2}$

**Correct option is (d)**

**Example 76.** One of the following is not an eigen state of the parity operator ? **[BHU 2016]**

(a) $\cos x + x\sin x$ (b) $x(\cos x + \sin x)$ (c) $x\cos x + \sin x$ (d) $x\cos x \sin x$

**Soln.** Now for $\psi(x) = x(\cos x + \sin x)$, we have $\hat{p}\psi(x) = -x(\cos x - \sin x) \neq \psi(-x)$ ;

So, $\psi(x) = x(\cos x + \sin x)$ is not an eigen state of the parity operator.

**Correct option is (b)**

**Example 77.** The wave function of a particle $\psi = \frac{1}{\sqrt{2}}\varphi_0 + i\varphi_1$ is given in terms of the eigen states $\varphi_0$ and $\varphi_1$ corresponding to the ground state energy and the first excited state energy $E_0$ and $E_1$ respectively. The expectation value of the Hamiltonian in the state $\psi$ is given by: **[BHU 2016]**

(a) $\frac{1}{2}E_0 + E_1$ (b) $\frac{E_0}{2} - E_1$ (c) $\frac{E_0 - 2E_1}{3}$ (d) $\frac{E_0 + 2E_1}{3}$

**Soln.** Given : $\psi = \frac{1}{\sqrt{2}}\phi_0 + i\phi_1$

$$\therefore \langle H\rangle = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\left\langle \frac{1}{\sqrt{2}}\phi_0 + i\phi_1 \middle| H \middle| \frac{1}{\sqrt{2}}\phi_0 + i\phi_1 \right\rangle}{\left\langle \frac{1}{\sqrt{2}}\phi_0 + i\phi_1 \middle| \frac{1}{\sqrt{2}}\phi_0 + i\phi_1 \right\rangle}$$

$$= \frac{\frac{1}{2}\langle\phi_0|H|\phi_0\rangle + \langle\phi_1|H|\phi_1\rangle}{\frac{1}{2}\langle\phi_0|\phi_0\rangle + \langle\phi_1 + \phi_1\rangle} = \frac{\frac{1}{2}E_0 + E_1}{\frac{1}{2} + 1} = \frac{E_0 + 2E_1}{3}$$

**Correct option is (d)**

**Example 78.** If H is a Hermitian matrix, then $e^{iH}$ is one of the following : **[BHU 2016]**

(a) Hermitian matrix (b) Unitary matrix (c) Orthogonal matrix (d) Null matrix

**Soln.** $\left(e^{iH}\right)^{\dagger} = e^{-iH^{\dagger}} = e^{-iH}$

Therefore, $\left(e^{iH}\right)\left(e^{iH}\right)^{\dagger} = e^{iH}e^{-iH} = I$ i.e it is a unitary matrix.

**Correct option is (b)**

**Example 79.** According to wave mechanism, a free particle in space can possess energies of the following type : **[HCU 2010]**

(a) Discrete energies (b) Continuous energies

(c) Both types of energies (d) Only one single value of energy

**Soln.** A free particle can possess any value of the energy.

**Correct option is (b)**

**Example 80.** The commutator of two hermitian matrices is

(a) hermitian (b) unitary (c) anti-hermitian (d) orthogonal **[HCU 2011]**

**Soln.** If $A$ and $B$ are two Hermitian matrices, then $A^{\dagger} = A$ and $B^{\dagger} = B$.

Now, $[A, B]^{\dagger} = (AB - BA)^{\dagger} = B^{\dagger}A^{\dagger} - A^{\dagger}B^{\dagger} = BA - AB$

$\Rightarrow \quad [A, B]^{\dagger} = -(AB - BA) = -[A, B]^{\dagger}$

So the commutator of two Hermitian matrices is an anti-Hermitian matrix.

**Correct option is (c)**

**Example 81.** The energy eigenstate of a system must necessarily be a **[HCU 2013]**

(a) A stationary state (b) A state having explicit time dependence

(c) A state having implicit time dependence (d) A state with no time dependence

**Soln.** Energy eigenstate of a system will have an explicit time dependency.

**Correct option is (b)**

**Example 82.** If A and B are hermitian matrices then $[A, B]$ is **[HCU 2014]**

(a) hermitian (b) anti-hermitian (c) unitary (d) symmetric

**Soln.** If $A$ and $B$ are two Hermitian matrices, then $A^{\dagger} = A$ and $B^{\dagger} = B$.

Now, $[A, B]^{\dagger} = (AB - BA)^{\dagger} = B^{\dagger}A^{\dagger} - A^{\dagger}B^{\dagger} = BA - AB$

$\Rightarrow \quad [A, B]^{\dagger} = -(AB - BA) = -[A, B]^{\dagger}$

So the commutator of two Hermitian matrices is an anti-Hermitian matrix.

**Correct option is (b)**

**Example 83.** Which of the following functions can be an eigen-function of the operator $i\hbar\frac{d}{dx}$?

(a) $e^{ikx}$ (b) $\cos(kx)$ (c) $\sin(kx)$ (d) $\cos(2kx)$ **[HCU 2014]**

**Soln.** If $\psi$ is the eigen function of the operator $\hat{A}$ with eigenvalue $a$, then $\hat{A}|\psi\rangle = a|\psi\rangle$

Here, $i\hbar\dfrac{d}{dx}e^{ikx}=-k\hbar\, e^{ikx}=a\,e^{ikx}$

**Correct option is (a)**

**Example 84.** If $\psi_1$ and $\psi_2$ are two independent solutions of the time-independent Schrodinger equation, which of the following is also a solution of the same Schrodinger equation? **[HCU 2015]**

(a) $\psi_1\times\psi_2$ (b) $\psi_1/\psi_2$ (c) $\psi_1\pm\psi_2$ (d) $(\psi_1\psi_2)^{1/2}$

**Soln.** If $\psi_1$ and $\psi_2$ are the wave function then according to superposition theory, the wave function of the composite system will be $\psi=(\psi_1\psi_2)$ and energy will be $E=E_1+E_2$.

**Correct option is (a)**

**Example 85.** Which of the following functions is the eigen-function of the operator, $-i\hbar\dfrac{d}{dx}$?

(a) $\exp(ikx)$ (b) $\cos(kx)$ (c) $\sin(kx)$ (d) $(\cos(kx)+\sin(kx))$

**Soln.** If $\psi$ is the eigen function of the operator $\hat{A}$ with eigenvalue $a$, then $\hat{A}|\psi\rangle=a|\psi\rangle$ **[HCU 2015]**

Here, $-i\hbar\dfrac{d}{dx}e^{ikx}=k\hbar\, e^{ikx}=a\,e^{ikx}$

**Correct option is (a)**

**Example 86.** In a one dimensional problem, the normalized wave function for the ground state is given by $\psi=N\,e^{-ax^2}$; where $N$ is the normalization constant. What is the value of $N$? **[HCU 2015]**

(a) $\sqrt{\dfrac{a}{\pi}}$ (b) $\sqrt{\dfrac{a}{2\pi}}$ (c) $\sqrt{\dfrac{2a}{\pi}}$ (d) $\dfrac{a}{\pi}$

**Soln.** $\psi=N\,e^{-ax^2}$

According to normalization condition

$$\int_{-\infty}^{\infty}|\psi|^2\,dx=1$$

$$\Rightarrow |N|^2\int_{-\infty}^{\infty}e^{-2ax^2}dx=1\Rightarrow 2|N|^2\int_{0}^{\infty}e^{-2ax^2}dx=1 \qquad \left[2ax^2=z\Rightarrow dx=\frac{dz}{2\sqrt{2az}}\right]$$

$$\Rightarrow \frac{1}{2}\sqrt{\frac{2}{a}}|N|^2\int_0^{\infty}e^{-z}z^{-1/2}\,dz=1\Rightarrow \frac{1}{2}\sqrt{\frac{2}{a}}|N|^2\sqrt{\pi}=1$$

$$\Rightarrow |N|^2=\sqrt{\frac{2a}{\pi}}\Rightarrow |N|=\left(\frac{2a}{\pi}\right)^{1/4}$$

**Correct option is ()**

**Example 87.** Commutator of two Hermitian operators has to be **[HCU 2015]**

(a) Hermitian (b) anti-Hermitian (c) unitary (d) orthogonal

**Soln.** If $A$ and $B$ are two Hermitian matrices, then $A^\dagger = A$ and $B^\dagger = B$.

$$\text{Now, } [A, B]^\dagger = (AB - BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = BA - AB$$

$$\Rightarrow \quad [A, B]^\dagger = -(AB - BA) = -[A, B]^\dagger$$

So the commutator of two Hermitian matrices is an anti-Hermitian matrix.

**Correct option is (b)**

**Example 88.** If $\psi_0(x) = u_0(x)$ and $\psi_1(x) = u_1(x) + \alpha\psi_0(x)$ for $a \le x \le b$ then $\psi_1(x)$ will be orthogonal to $\psi_0(x)$ if **[HCU 2016]**

(a) $\alpha = -\frac{1}{|\psi_0|^2}\int_a^b \psi_0^* u_1 dx$ (b) $\alpha = \frac{1}{|\psi_0|^2}\int_a^b \psi_0^* u_1 dx$

(c) $\alpha = -\frac{1}{|\psi_0|^2}\int_a^b \psi_0^* u_0 dx$ (d) $\alpha = \frac{1}{|\psi_0|^2}\int_a^b \psi_0^* u_0 dx$

**Soln.** $\psi_0(x) = u_0(x)$

$\psi_1(x) = u_1(x) + \alpha\psi_0(x) = u_1(x) + \alpha u_0(x)$

$\psi_0(x)$ and $\psi_1(x)$ will be orthogonal when

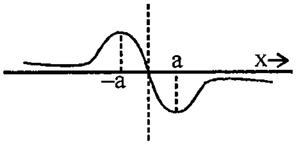
$$\langle \psi_0 | \psi_1 \rangle = 0 \Rightarrow \int_a^b \psi_0(x)\,\psi_1(x)\,dx = 0$$

$$\Rightarrow \quad \int_a^b \left(u_0^*(x)u_1(x) + |u_0(x)|^2\right)dx = 0 \Rightarrow \int_a^b u_0^*(x)u_1(x)\,dx + \alpha\int_a^b |u_0(x)|^2 dx = 0$$

$$\Rightarrow \quad \alpha = -\frac{1}{\int_a^b |\psi_0|^2 dx}\int_a^b \psi_0^*(x)\,u_1(x)\,dx$$

**Correct option is ( )**

**Example 89.** The wave function of the first excited state of a particle in a simple harmonic oscillator potential may be represented as shown in the figure below: **[DU 2014]**



The probability of finding the particle at a particular position is maximum for:

(a) only x = a (b) only x = –a

(c) Both x = a and x = –a (d) x tending to ±∞

**Soln.** The probability would be maximum where $|\psi(x)|^2$ is maximum and $|\psi(x)|^2$ is maximum makes $|\psi(x)|$ also either maximum or minimum. So the probability in our case is maximum at $x = \pm a$.

**Correct option is (c)**

**Example 90.** Which of the following is an admissible wave function in three dimensions? (Here $N_1$, $N_2$, $N_3$, $N_4$ and $r_0$ are constants and r is the radial coordinate): **[DU 2014]**

(a) $\psi = \frac{N_1}{(r^2 + r_0^2)^{1/4}}$

(b) $\psi = \frac{N_2}{(r^2 + r_0^2)^{1/2}}$

(c) $\psi = \frac{N_3}{(r^2 + r_0^2)^{3/2}}$

(d) $\psi = \frac{N_4}{(r^2 + r_0^2)^{2}}$

**Soln.** Wave function should be finite anf sigle valued everywhere.

**Correct option is (d)**

**Example 91.** The wave function of a particle is given by $\psi = \frac{1}{\sqrt{2}} e^{-|x|/2}$ with $-\infty < x < +\infty$. The minimum uncertainty in the momentum is: **[DU 2014]**

(a) $10h/(2\pi)$ (b) $h/(2\pi\sqrt{2})$ (c) $2/h$ (d) $\sqrt{h}$

**Soln.** Given : $\psi(x) = \frac{1}{\sqrt{2}} e^{-|x|/2} = \begin{cases} \frac{1}{\sqrt{2}} e^{-x/2} & ; \quad x \geq 0 \\ \frac{1}{\sqrt{2}} e^{x/2} & ; \quad x < 0 \end{cases}$

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi(x) x \,\psi(x)\, dx \qquad = \frac{1}{2}\left[\int_{-\infty}^{0} x e^{+x/2} dx + \int_{0}^{\infty} x e^{-x} dx\right]$$

$$= \frac{1}{2}\left[\left\{x\left(\frac{e^{+x}}{+1}\right) - \left(\frac{e^{+x}}{+1}\right)\right\}_{-\infty}^{0} + \left\{x\left(\frac{e^{-x}}{-1}\right) - \left(\frac{e^{-x}}{1}\right)\right\}_{0}^{\infty}\right] = \frac{1}{2}[(-1) - (0) + (0) - (-1)] \qquad = 0.$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi(x) x^2 \,\psi(x)\, dx \qquad = \frac{1}{2}\left[\int_{-\infty}^{0} x^2 e^{x} dx + \int_{0}^{\infty} x^2 e^{-x} dx\right]$$

$$= \frac{1}{2}\left[\left\{x^2(e^x) - 2xe^x + 2e^x\right\}_{-\infty}^{0} + \left\{x^2\left(\frac{e^{-x}}{-1}\right) - 2x\left(\frac{e^{-x}}{1}\right) + 2\left(\frac{e^{-x}}{-1}\right)\right\}_{0}^{\infty}\right]$$

$$= \frac{1}{2}[(2) - (0) + (0) - (-2)] \qquad = 2.$$

$$\therefore \quad \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{2}$$

$$\therefore \quad (\Delta p)_{\min} = \frac{h}{4\pi\,\Delta x} = \frac{h}{4\sqrt{2}\pi}.$$

**Correct option is ( ).**

**Example 92.** The wave function of the particle is given by a linear combination of two of its energy states $\psi_1$ and $\psi_2$ (with energy eigen values $E_1$ and $E_2$ respectively) as $\psi = \psi_1 / a + \psi_2 / b$. If a measurement of energy is made, the probability of finding particle energy to be $E_1$ is **[DU 2015]**

(a) $|b|^2 /[|a|^2 + |b|^2]$ (b) $|a|^2 /[|a|^2 + |b|^2]$

(c) $b/(a+b)$ (d) $a/(a+b)$

**Soln.** Probability of finding particle energy to be $E_1$ is

$$P(E_1) = \frac{|\langle \psi_1 | \psi \rangle|^2}{\langle \psi | \psi \rangle} = \frac{\left|\left\langle \psi_1 \middle| \frac{\psi_1}{a} + \frac{\psi_2}{b} \right\rangle\right|^2}{\left\langle \frac{\psi_1}{a} + \frac{\psi_2}{b} \middle| \frac{\psi_1}{a} + \frac{\psi_2}{b} \right\rangle} = \frac{1/|a|^2}{1/|a|^2 + 1/|b|^2} = \frac{|b|^2}{|a|^2 + |b|^2}$$

**Correct option is (a)**

**Example 93.** The wave function for a particle is given by $\psi = Ae^{-|x|}$, where A is a constant. This implies that the potential

(a) is differentiable everywhere **[DU 2015]**

(b) is continuous everywhere but not differentiable at $x = 0$

(c) has discontinuity at $x = 0$

(d) has discontinuities at $x = \pm 1$ but differentiable every else

**Soln.** Given : $\psi = Ae^{-|x|} = \begin{cases} Ae^{-x} & ; \quad x > 0 \\ Ae^{x} & ; \quad x < 0 \end{cases}$

$$\Rightarrow \frac{d\psi}{dx} = \begin{cases} -Ae^{-x} & ; \quad x > 0 \\ Ae^{x} & ; \quad x < 0 \end{cases}$$

So at $x = 0$, $\frac{d\psi}{dx}$ is not continuous and hence not differentiable as well. This is because the potential at $x = 0$ is not continuous as well.

**Correct option is (c)**

# Chapter 5

# Bound State problems

## 5.1. What is a Bound State?

If the motion of a particle is confined to a limited region of space by potential energy so that the particle can move back and forth in the region, then the particle is in a bound state. The simplest example of all motions in bound state is the motion of a particle in a one-dimensional box with zero potential energy and the potential energy is assumed to be infinite at he walls of the box.

The motion of a particle in three-dimensional box, one-dimensional box, motion of a simple harmonic oscillator, motion of a particle in a one-dimensional square well potential with ($E < V_0$), motion of a particle in a one-dimensional square well potential with ($E < V_0$), motion of a particle in one-dimensional (also in three-dimensional) potential well, motion of a particle in spherically symmetric potential, etc. are the common examples of bound state. Application of Schrodinger time-independent equal to such problems leads to discrete energy values of the particles.

## 5.2. Particle in a One-Dimensional Box:

Consider a particle of mass '*m*' and energy '*E*' is moving along x-axis, in the region from $x = 0$ to $x = a$ under the following potential i.e.

$$V(x) = 0 \qquad 0 < x < a$$
$$= \infty \qquad \text{otherwise}$$

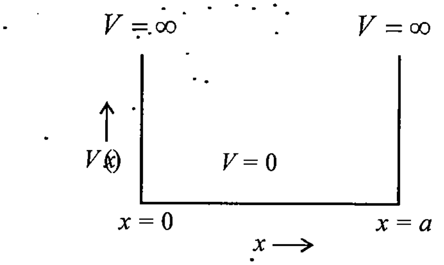


**Figure :** Schematic diagram of 1-D infinite potential well.

The particle experiences no force within the potential well but feels a sudden large force directed towards the origin as it reaches the points $x = 0, a$. Since, the potential is infinite outside, the particle cannot penetrate outside of the region $0 < x < a$ i.e. $\psi = 0$ for the particle outside the box.

According to 1-D time-independent Schrodinger equaion,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V\psi(x) = E\psi(x) \qquad (1)$$

For the region $0 < x < a$, the equation becomes,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \qquad \Rightarrow \frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0$$

$$\Rightarrow \frac{d^2\psi}{dx^2} + k^2x = 0 \text{ where } k^2 = \frac{2mE}{\hbar^2} \tag{2}$$

Solution of eqn. (2) can be written as,

$$\psi(x) = A\sin kx + B\cos kx \tag{3}$$

Since, the particle remains confined within the box, then the probablity of finding the particle at $x = 0$ and $x = a$ must be zero.

Boundary conditions: $\psi(x)$ will be continuous at $x = 0$ and $x = a$

(i) Applying $\psi(x)\big|_{x=0} = 0 \qquad \Rightarrow B = 0$

(ii) Applying $\psi(x)\big|_{x=a} = 0 \qquad \Rightarrow A\sin ka = 0 \qquad \Rightarrow ka = n\pi \;\Rightarrow k = \frac{n\pi}{a}$

$$\Rightarrow \psi(x) = A\sin\left(\frac{n\pi x}{a}\right)(n = 1, 2, 3, \dots\dots\dots) \text{ for } 0 < x < a \tag{4}$$

**Note:** If $n = 0 \;\Rightarrow\; k = 0 \Rightarrow E = 0 \Rightarrow\; \psi(x) = 0$ everywhere inside the box. Thereofore, there will be no admissible particle with zero energy within the box.

**Normalization of the wave function:**

The wave function $\psi_n$ corresponding to n<sup>th</sup> quantum state is

$$\psi_n(x) = A\sin\frac{n\pi x}{a} \qquad 0 \le x \le a$$
$$= 0 \qquad \text{otherwise}$$

Since, the particle must be somewhere within the box, the total probability of finding the particle inside the box is unity i.e.

$$\int_0^a \psi_n^*(x)\psi_n(x)\,dx = 1 \qquad \Rightarrow \int_0^a A^2\sin^2\frac{n\pi x}{a}dx = 1$$

$$\Rightarrow \quad A^2\int_0^a \frac{1}{2}\left[1 - \cos\frac{2\pi nx}{a}\right]dx = 1 \;\Rightarrow\; A = \sqrt{\frac{2}{a}}$$

Therefore, the normallized wave function will be

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \qquad (n = 1, 2, 3, \dots\dots\dots\dots\dots)$$

**Energy eigenvalues:**

We know that, $\qquad k = \sqrt{\frac{2mE}{\hbar^2}} \qquad \Rightarrow \frac{n^2\pi^2}{a^2} = \frac{2mE}{\hbar^2} \qquad \Rightarrow E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \qquad (n = 1, 2, 3, \dots\dots)$

So, we get an infinite sequence of descrete energy levels that corresponds to all integral values of $n$, where $n$ is called the quantum number representing the different states of the particle.

Ground state energy: $n = 1 \quad \Rightarrow E_1 = \frac{\pi^2\hbar^2}{2ma^2}$

First excited state energy: $n = 2 \quad \Rightarrow E_2 = \dfrac{4\pi^2\hbar^2}{2ma^2}$

In general, $\quad E_n = n^2 E_1$

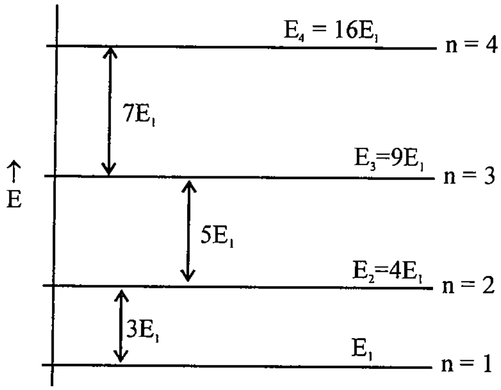


**Figure :** Energy levels of a particle in a 1-D infinite potential well

Difference between two consecutive energy levels

$$\Delta E_n = E_{n+1} - E_n = \frac{\pi^2\hbar^2}{2ma^2}\left[(n+1)^2 - n^2\right] = \frac{\pi^2\hbar^2}{2ma^2}(2n+1)$$

Spacing between two energy levels increases with 'n', decreases with size of the box 'a' and the mass of the particle 'm'. As $a \to \infty \Rightarrow \Delta E_n = 0$ i.e. the energy levels gets closer, forming a continuum of energy.

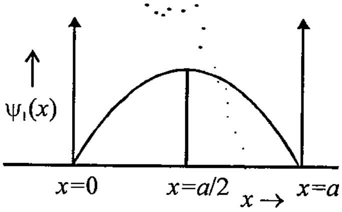
**Momentum eigenvalues:**

$$E_n = \frac{p_n^2}{2m} \qquad \Rightarrow P_n^2 = 2mE_n = 2m\frac{n^2\pi^2\hbar^2}{2ma^2} = \frac{n^2\pi^2\hbar^2}{a^2} \qquad \Rightarrow P_n = \pm\frac{n\pi\hbar}{a}$$

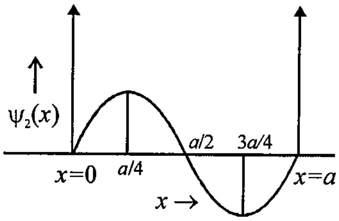
where '$\pm$' sign is due to the back and forth motion of the particle within the box.
Difference of momentum between two consecutive energy levels

$$\Delta P_n = P_{n+1} - P_n = \pm\left[\frac{(n+1)\pi\hbar}{a} - \frac{n\pi\hbar}{a}\right] = \pm\frac{\pi\hbar}{a}$$

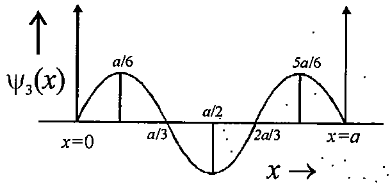
**Various eigenstates of the particle inside a 1-D box :**



Ground state



First excited state



**Note:** $\psi_n(x)$ has $(n-1)$ nodes (excluding the boundary at $x = 0$ and $x = a$)

**Variation of probability density corresponding to various eigenstates of the particle:**

Probability density of finding the particle in the nth quantum state is given by,

$$P_n(x)=|\psi_n(x)|^2=\frac{2}{a}\sin^2\frac{n\pi x}{a}$$

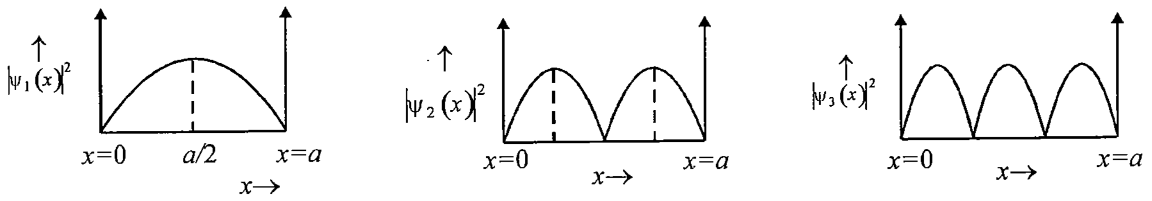


**Figure :** Variation of probability density corresponding to ground, first excited and second excited state respectively

**Note:**

(1) Probability of finding the particle in the left half of the well = Probability of finding the particle in the left half of the well = 1/2, for any state *n*.

(2) Probability of finding the particle between $\frac{a}{4}<x<\frac{3a}{4}$ is 1/2 for $n = 2, 4, 6, .......$

- The energy eigenfunctions of the particle confined in a 1-D box, satisfies the orthonormality condition i.e.

$$\int_0^a \psi_m^*(x)\psi_n(x)\,dx=\delta_{mn}$$

- As the energy eigenfunctions form a complete set, an arbitary function $\psi(x)$ (which is well behaved and satisfy the same boundary conditions) can be expanded in terms of these eigenfunctions as

$$\psi(x)=\sum_{n=1}^{\infty}c_n\,\psi_n(x)=\sum_{n=1}^{\infty}c_n\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$$

**Expectation values of $\hat{x},\hat{p}_x,\hat{x}^2,\hat{p}_x^2$:**

$$\langle x\rangle=\int_0^a\psi^*x\psi\,dx=\int_0^a\frac{2}{a}x\sin^2\left(\frac{n\pi x}{a}\right)dx=\frac{2}{a}\int_0^a\frac{x}{2}\left[1-\cos\frac{(2n\pi x)}{a}\right]dx$$

$$=\frac{2}{a}\frac{1}{2}\left[\frac{x^2}{2}-x\frac{\sin\frac{2\pi nx}{a}}{\frac{2\pi n}{a}}-\cos\frac{2n\pi x}{a}\right]_0^a=\frac{1}{a}\left[\frac{a^2}{2}-1+1\right]=\frac{a}{2}$$

$$\langle p\rangle=\int_0^a\psi^*\left(-i\hbar\frac{\partial}{\partial x}\right)\psi\,dx=\frac{2}{a}(-i\hbar)\int_0^a\sin\left(\frac{n\pi x}{a}\right)\frac{n\pi}{a}\cos\left(\frac{n\pi x}{a}\right)dx=0$$

**Note:** Expectation value of momentum for a real wave function is always zero.

$$\langle x^2\rangle=\int_0^a\psi^*x^2\,\psi\,dx=\frac{2}{a}\int_0^a x^2\sin^2\frac{n\pi x}{a}dx$$

$$=\frac{1}{a}\left[\left(\frac{x^3}{3}\right)_0^a-\left(x^2\frac{\sin 2n\pi x/a}{2n\pi/a}\right)_0^a+\int_0^a 2x.\sin\frac{2n\pi x/a}{2n\pi/a}dx\right]$$

$$=\frac{1}{a}\left[\frac{a^3}{3}-0+2\left\{-x\frac{\cos\frac{2n\pi x}{a}}{\left(\frac{2n\pi}{a}\right)^2}+\frac{\sin\frac{2n\pi x}{a}}{\left(\frac{2n\pi}{a}\right)^2}\right\}_0^a\right]=\frac{1}{a}\left[\frac{a^3}{3}-\frac{a^3}{2(n\pi)^2}\right]=\frac{a^2}{3}-\frac{a^2}{2n^2\pi^2}$$

$$\langle p^2\rangle=\frac{2}{a}\int_0^a\sin\left(\frac{n\pi x}{a}\right)\left(-\hbar^2\frac{\partial^2}{\partial x^2}\right)\sin\left(\frac{n\pi x}{a}\right)dx=\frac{2}{a}(-\hbar^2)\left(\frac{n\pi}{a}\right)^2(-1)\int_0^a\sin^2\left(\frac{n\pi x}{a}\right)dx$$

$$=\frac{2}{a}(-\hbar^2)\left(\frac{n\pi}{a}\right)^2(-1)\int_0^a\sin^2\left(\frac{n\pi x}{a}\right)dx.$$

$$=\frac{n^2\pi^2\hbar^2}{a^2}\int_0^a\left[1-\cos\left(\frac{2n\pi x}{a}\right)\right]dx=\frac{n^2\pi^2\hbar^2}{a^3}\left[x-\frac{\sin\frac{2n\pi x}{a}}{\frac{2n\pi}{a}}\right]_0^a=\frac{n^2\pi^2\hbar^2}{a^2}$$

Uncertainity in the position of the particle is

$$\Delta x=\left[\langle x^2\rangle-\langle x\rangle^2\right]^{1/2}=\left[\frac{a^2}{3}-\frac{a^2}{2n^2\pi^2}-\frac{a^2}{4}\right]^{1/2}=\left[\frac{a^2}{12}-\frac{a^2}{2n^2\pi^2}\right]^{1/2}$$

Uncertainity in the momentum of the particle is

$$\Delta p=\left[\langle p^2\rangle-\langle p\rangle^2\right]^{1/2}=\frac{n\pi\hbar}{a}$$

Therefore, the uncertainity product $\Delta x\Delta p=n\pi\hbar\left[\frac{1}{12}-\frac{1}{2n^2\pi^2}\right]^{1/2}$

**Example 1:** A particle in a deep square well potential extending from $x=0$ to $x=L$ has a wave function $\psi(x)=\frac{1}{\sqrt{5}}\left[|\phi_1\rangle+2|\phi_2\rangle\right]$ where $|\phi_1\rangle$ and $|\phi_2\rangle$ denote the ground state and the first excited state wave functions. Find the expectation of '$x$' in this state.

**Soln.** $\langle x\rangle=\langle\psi|x|\psi\rangle=\frac{1}{\sqrt{5}}\left(\left(\langle\phi_1|+2\langle\phi_2|\right)x\frac{1}{\sqrt{5}}\left(|\phi_1\rangle+2|\phi_2\rangle\right)\right)$

$$=\frac{1}{5}\left[\langle\phi_1|x|\phi_1\rangle+2\langle\phi_1|x|\phi_2\rangle+2\langle\phi_2|x|\phi_1\rangle+4\langle\phi_2|x|\phi_2\rangle\right]$$

Now, $\langle\phi_1|X|\phi_2\rangle=\int_0^L\left(\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}\right)x\left(\sqrt{\frac{2}{L}}\sin\frac{2\pi x}{L}\right)dx$

$$=\frac{1}{L}\int_0^L x\left(\cos\frac{\pi x}{L}-\cos\frac{3\pi x}{L}\right)dx=\frac{1}{L}\left(-\frac{2L^2}{\pi^2}+\frac{2L^2}{9\pi^2}\right)=-\frac{16L}{9\pi^2}$$

Therefore, $\langle x\rangle=\frac{1}{5}\left[\frac{L}{2}-\frac{32L}{9\pi^2}-\frac{32L}{9\pi^2}+4.\frac{L}{2}\right]=\frac{L}{2}-\frac{64L}{45\pi^2}$

**Example 2:** Calculate $\langle p_x \rangle$ for $\psi(x)=\frac{1}{\sqrt{2}}(|\phi_1\rangle+|\phi_2\rangle)$ where $|\phi_1\rangle$ and $|\phi_2\rangle$ are the ground state and the first excited state wave functions of a particle in a deep square well potential.

**Soln.** $\langle p_x\rangle=\langle\psi|p_x|\psi\rangle=\frac{1}{2}\left[\langle\phi_1|p_x|\phi_1\rangle+\langle\phi_2|p_x|\phi_1\rangle+\langle\phi_1|p_x|\phi_2\rangle+\langle\phi_2|p_x|\phi_2\rangle\right]$

$$\langle\phi_2|p_x|\phi_1\rangle=\int_{-\infty}^{\infty}\phi_2^*\left(-i\hbar\frac{d}{dx}\right)\phi_1(x)\,dx=\frac{2}{L}(-i\hbar)\int_0^L\sin\frac{2\pi x}{L}\frac{d}{dx}\left(\sin\frac{\pi x}{L}\right)dx$$

$$=\frac{-i\hbar\pi}{L^2}\int_0^L 2\sin\frac{2\pi x}{L}\cos\frac{\pi x}{L}\,dx=\frac{-i\hbar\pi}{L^2}\int_0^L\left(\sin\frac{3\pi x}{L}+\sin\frac{\pi x}{L}\right)dx=-\frac{8i\hbar}{3L}$$

Similarly, $\langle\phi_1|p_x|\phi_2\rangle=\frac{8i\hbar}{3L}$

Therefore, $\langle p_x\rangle=0$

**Example 3:** The wave function of a particle in a deep square well potential extending from $x=0$ to $x=L$ is $\psi(x)=\sqrt{30}\,x(x-L)/L^i$ (a) Find the value of i, (b) Find the average value of 'x' in this state.

**Soln.** (a) Any wave function of a particle in one dimension has the dimensions of $\frac{1}{\sqrt{L}}$. Thus $i=\frac{5}{2}$

(b) $\langle x\rangle=\int_{-\infty}^{\infty}\psi^*(x)x\psi(x)\,dx=\frac{30}{L^5}\int_0^L x^3(x-L)^2\,dx=\frac{L}{2}$

**Example 4:** A particle is in the ground state of a deep square well in the range $0<x<L$. Suddenly the wall $x=L$ of the well is shifted to $x=2L$, so as to make it a deep square well in the range $0<x<2L$. Immediately after this, the energy of the particle is measured. What will be the probability of getting the energy as the new ground state energy?

**Soln.** The wave function just before the shifting of the wall is

$$\psi(x)=\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}\ \text{for } 0<x<L$$
$$=0\qquad\text{otherwise}$$

This is the energy eigenfunction corresponding to the ground state in the original well. When the wall is shifted, the new ground state wave function becomes,

$$\phi(x)=\sqrt{\frac{2}{2L}}\sin\frac{\pi x}{2L}\ \text{for } 0<x<2L$$
$$=0\qquad\text{otherwise}$$

The probability of getting the ground state energy in measurements is $|\langle\phi|\psi\rangle|^2$

$$\langle\phi|\psi\rangle=\int_{-\infty}^{\infty}\phi^*(x)\psi(x)\,dx=\int_0^L\sqrt{\frac{2}{2L}}\sin\frac{\pi x}{2L}\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}\,dx$$

$$=\frac{1}{L\sqrt{2}}\int_0^L\left(\cos\frac{\pi x}{2L}-\cos\frac{3\pi x}{2L}\right)dx=\frac{1}{L\sqrt{2}}\left|\frac{\sin\frac{\pi x}{2L}}{\frac{\pi}{2L}}-\frac{\sin\frac{3\pi x}{2L}}{\frac{3\pi}{2L}}\right|_0^L=\frac{8}{3\sqrt{2}\,\pi}$$

Therefore, the probability is $|\langle\phi|\psi\rangle|^2=\dfrac{32}{9\pi^2}$

**Example 5:** Consider a particle of mass $10^{-30}$ kg trapped in an infinitely deep potential well of width 0.5 nm. (a) Find the value of the wave number 'k' in the ground state. (b) How many states are there having value of 'k' between 10 $nm^{-1}$ and 100 $nm^{-1}$?

**Soln.** (a) $k=\dfrac{n\pi}{L}$. In ground state, $k=\dfrac{\pi}{L}=\dfrac{3.14}{0.5nm}=6.28\ nm^{-1}$

(b) For every interval $\dfrac{\pi}{L}$ of k, there is one quantum state.

The total k-interval is, $\Delta k=100\ nm^{-1}-10\ nm^{-1}=90\ nm^{-1}$

So, the number of states with 'k' value in the given interval is, $\dfrac{90\ nm^{-1}}{6.28\ nm^{-1}}=14.33$

There are 14 states in the given interval.

**Example 6:** A particle of mass 'm' which moves freely inside an infinite potential well of width 'a' has the following wave fn (at $t=0$)

$$\psi(x,0)=\frac{A}{\sqrt{a}}\sin\left(\frac{\pi x}{a}\right)+\sqrt{\frac{3}{5a}}\sin\left(\frac{3\pi x}{a}\right)+\frac{1}{\sqrt{5a}}\sin\left(\frac{5\pi x}{a}\right)\quad[A=\text{real constant}]$$

(a) Find the value of A such that $\psi(x,0)$ is normalized to unity.

(b) If the measurement is carried out, what values will be found and with what probability? and also calculate $\langle E\rangle$

(c) Find the wavefunction $\psi(x,t)$

(d) Determine the probability of finding the system at time 't' in the state $\phi(x,t)=\sqrt{\dfrac{2}{a}}\sin\left(\dfrac{5\pi x}{a}\right)e^{-iE_5\frac{t}{\hbar}}$

**Soln.** (a) $$\psi(x,0)=\frac{A}{\sqrt{2}}\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)+\sqrt{\frac{3}{10}}\sqrt{\frac{2}{a}}\sin\left(\frac{3\pi x}{a}\right)+\frac{1}{\sqrt{10}}\sqrt{\frac{2}{a}}\sin\left(\frac{5\pi x}{a}\right)$$

$$\Rightarrow\quad\psi(x,0)=\frac{A}{\sqrt{2}}|\phi_1\rangle+\sqrt{\frac{3}{10}}|\phi_3\rangle+\frac{1}{\sqrt{10}}|\phi_5\rangle$$

Normalization condition: $\langle\psi(x,0)|\psi(x,0)\rangle=1\ \Rightarrow\ \sum\limits_{n=1,3,5}|C_n|^2=1$

$$\Rightarrow\frac{A^2}{2}+\frac{3}{10}+\frac{1}{10}=1\ \Rightarrow\ A^2=\frac{6}{5}\ \Rightarrow A=\sqrt{\frac{6}{5}}$$

Therefore, $$|\psi(x,0)\rangle=\sqrt{\frac{6}{10}}|\phi_1\rangle+\sqrt{\frac{3}{10}}|\phi_3\rangle+\sqrt{\frac{1}{10}}|\phi_5\rangle$$

(b) The state of the particle is a linear combination of ground state $|\phi_1\rangle$, 2nd excited state $|\phi_3\rangle$ and 4th excited state $|\phi_5\rangle$ respectively. Therefore, the energies we will get are

$$E_1=\frac{\pi^2\hbar^2}{2ma^2},E_3=\frac{9\pi^2\hbar^2}{2ma^2},E_5=\frac{25\pi^2\hbar^2}{2ma^2}$$

Now, the respective probabilties are

$$P(E_1)=|\langle\phi_1|\psi\rangle|^2=\frac{3}{5};\qquad P(E_2)=|\langle\phi_2|\psi\rangle|^2=\frac{3}{10};\qquad P(E_1)=|\langle\phi_3|\psi\rangle|^2=\frac{1}{10}$$

$$\begin{aligned}\langle E\rangle &= P(E_1).E_1+P(E_3).E_3+P(E_5).E_5\\ &=\frac{3}{5}.\frac{\pi^2\hbar^2}{2ma^2}+\frac{3}{10}\frac{9\pi^2\hbar^2}{2ma^2}+\frac{1}{10}\frac{25\pi^2\hbar^2}{2ma^2}\\ &=\frac{\pi^2\hbar^2}{2ma^2}\left[\frac{3}{5}+\frac{27}{10}+\frac{5}{2}\right]=\frac{29\pi^2\hbar^2}{10ma^2}\end{aligned}$$

(c) $|\psi(x,t)\rangle=\sqrt{\frac{6}{10}}|\phi_1\rangle e^{-iE_1 t/\hbar}+\sqrt{\frac{3}{10}}|\phi_3\rangle e^{-iE_3 t/\hbar}+\sqrt{\frac{1}{10}}|\phi_5\rangle e^{-iE_5 t/\hbar}$

(d) $|\phi(x,t)\rangle=|\phi_5\rangle e^{-iE_5 t/\hbar}\quad\Rightarrow\quad P=|\langle\phi(x,t)|\psi\rangle|^2=\frac{1}{10}$

**Example 7:** A particle of mass $m$ confined to move in a potential $V(x)=0$ for $0\le x\le a$ and $V(x)=\infty$ otherwise. The wave function of the particle at time $t=0$ is

$\psi(x,0)=A\left(2\sin\frac{\pi x}{a}+\sin\frac{3\pi x}{a}\right)$ (i) Normalize $\psi(x,0)$; (ii) find $\psi(x,t)$.

**Soln.** For a particle, in the potential given, the energy eigenvalues and eigenfunctions are given by

$$E_n=\frac{n^2\pi^2\hbar^2}{2ma^2},\ \psi_n(x)=\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a},\qquad n=1,\ 2,\ 3,\ \ldots.$$

(i) Normalization condition: $\int_0^a\psi^*\psi dx=1\Rightarrow A^2\int_0^a\left(2\sin\frac{\pi x}{a}+\sin\frac{3\pi x}{a}\right)^2dx=1$

$$\Rightarrow A^2\left(4\frac{a}{2}+\frac{a}{2}\right)=1\Rightarrow A=\sqrt{\frac{2}{5a}}$$

Therefore, $\psi(x,0)=\frac{1}{\sqrt{5}}\left(2\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}+\sqrt{\frac{2}{a}}\sin\frac{3\pi x}{a}\right)=\frac{1}{\sqrt{5}}(2\psi_1+\psi_3)$

(ii) $\psi(x,t)=\frac{1}{\sqrt{5}}(2\psi_1e^{-iE_1t/\hbar}+\psi_3e^{-iE_3t/\hbar})$

**Example 8:** A particle in a box is in a superposition state and is described by the wave function

$$\psi(x,t)=\frac{1}{\sqrt{a}}\left[\exp\left(\frac{-iE_1t}{\hbar}\right)\cos\frac{\pi x}{2a}+\exp\left(\frac{-iE_2t}{\hbar}\right)\sin\frac{2\pi x}{2a}\right],\qquad -a<x<a$$

where $E_1$ and $E_2$ are the energy eigenvalues of the first two states. Find the expectation value of $x$.

**Soln.** $\langle x\rangle=\int_{-\infty}^{\infty}\psi^*(x,t)x\psi(x,t)dx=\frac{1}{a}\int_{-a}^{a}x\cos^2\frac{\pi x}{2a}dx+\frac{1}{a}\int_{-a}^{a}x\sin^2\frac{2\pi x}{2a}dx$

$$+\frac{1}{a}\{\exp[i(E_1-E_2)t/\hbar)]+\exp[i(E_2-E_1)t/\hbar)]\}\int_{-a}^{a} x\cos\frac{\pi x}{2a}\sin\frac{2\pi x}{2a}dx$$

The integrands in the first two terms are odd and hence will not contribute.

$$\int_{-a}^{a} x\cos\frac{\pi x}{2a}\sin\frac{2\pi x}{2a}dx=\int_{-a}^{a}\frac{x}{2}\left(\sin\frac{3\pi x}{2a}+\sin\frac{\pi x}{2a}\right)dx=\frac{1}{2}\left[-\frac{8a^2}{9\pi^2}+\frac{8a^2}{\pi^2}\right]=\frac{32a^2}{9\pi^2}$$

Therefore,

$$\langle x\rangle=\frac{2}{a}\cos\frac{(E_2-E_1)t}{\hbar}\left(\frac{32a^2}{9\pi^2}\right)=\frac{64a}{9\pi^2}\cos\frac{(E_2-E_1)t}{\hbar}$$

**Example 9:** For an electron in a one-dimensional infinite potential well of width 1Å, calculate (i) the separation between the two lowest energy levels; (ii) the frequency and wavelength of the photon corresponding to a transition between these two levels; and (iii) in what region of the electromagnetic spectrum is this frequency / wavelength?

**Soln.** (i) Energy of the particle in the n^th^ state $E_n=\frac{\pi^2\hbar^2n^2}{2ma^2}$

Separation between two lowest enegy states will be

$$E_2-E_1=\frac{3\pi^2\hbar^2}{2ma^2}=113.27\text{eV}$$

(ii) Frequency of the photon $\nu=\frac{E}{h}=2.7\times10^{16}\ Hz$

Wavelength of the photon $\lambda=\frac{c}{\nu}=\frac{3\times10^8\text{ms}^{-1}}{2.7\times10^{16}\text{s}^{-1}}=1.1\times10^{-8}\text{m}$

(iii) This frequency falls in the vacuum ultraviolet region.

**Example 10:** A particle of mass *m* trapped in the potential $V(x)=0$ for $-a\le x\le a$ and $V(x)=\infty$ otherwise. Evaluate the probability of finding the trapped particle between $x=0$ and $x=a/n$ when it is in the *n*th state.

**Soln.** Wave function $\psi_n(x)=\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$

Required probability $P=\int_0^{a/n}\psi_n^*\psi_n dx=\frac{2}{a}\int_0^{a/n}\sin^2\frac{n\pi x}{a}dx\frac{1}{a}\int_0^{a/n}\left(1-\cos\frac{2n\pi x}{a}\right)dx=\frac{1}{n}$

**Example 11:** An electron is in the ground state of a one-dimensional infinite square well with $a=10^{-10}$ m. Compute the force that the electron exerts on the wall during an impact on either wall.

**Soln.** The force on the wall $F=-\frac{dE_n}{da}$

The energy of the ground state $E_1=\frac{\pi^2\hbar^2}{2ma^2}$

and hence the force on the wall $F = -\left.\frac{dE_1}{da}\right|_{a=10^{-10}m} = \left.\frac{\pi^2\hbar^2}{ma^3}\right|_{a=10^{-10}m} = 1.21\times10^{-7}N$

## 5.3. Particle in a 1-D Symmetric Infinite Potential Well

If a particle be constrained to move in a potential as below,

$$V(x) = \begin{cases} 0, & \text{for} -a/2 < x < a/2 \\ +\infty, & \text{for } |x| > a/2 \end{cases}$$

having perfectly rigid and impenetrable walls at $x = -a/2$ and $x = +a/2$, then also we can show that the energy eigenvalues are discrete similar to the previous case.

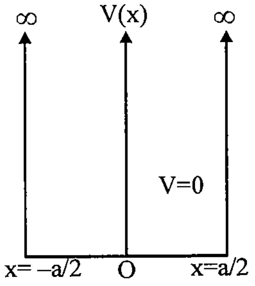


Fig.

The Schrodinger time independent equation for the particle in the region, $-a/2 < x < a/2$ is,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \text{ or, } \frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}E\psi = 0$$

where $m$ is the mass of the particle. The general solution of equation is, $\psi(x) = A\cos\alpha x + B\sin\alpha x$

where, $\alpha = \left(\frac{2mE}{\hbar^2}\right)^{1/2}$.

Here, A and B are two constants of integration which can be found out by applying the following boundary conditions:

**1.** $\psi = 0$ at $x = a/2$ and **2.** $\psi = 0$ at $x = -a/2$

Applying these boundary conditions to equation we obtain,

$$A\cos\frac{\alpha a}{2} + B\sin\frac{\alpha a}{2} = 0, \quad A\cos\frac{\alpha a}{2} - B\sin\frac{\alpha a}{2} = 0 \quad \therefore \quad A\cos\frac{\alpha a}{2} = 0 \text{ and } B\sin\frac{\alpha a}{2} = 0$$

Now since both A and B cannot be zero (as in that case $\psi(x) = 0$ everywhere). Also for given value of $\alpha$ or $E$, cos $\frac{\alpha a}{2}$ and sin $\frac{\alpha a}{2}$ cannot be zero.

There are two possible solutions, viz., either $B = 0$ and cos $\frac{\alpha a}{2} = 0$,

$\therefore \quad \frac{\alpha a}{2} = \frac{n\pi}{2}$ ($n$ being an odd integer) or, A = 0 and $\sin\frac{\alpha a}{2} = 0$,

$\therefore \quad \frac{\alpha a}{2} = \frac{n\pi}{2}$ ($n$ being an even integer)

So, $\quad \psi(x) = A\cos\frac{n\pi x}{a}$ (when $n$ is odd) and $\psi(x) = B\sin\frac{n\pi x}{a}$ (when $n$ is even)

The energy eigenvalue for the $n$th quantum state is given by,

$$\frac{\alpha a}{2} = \frac{n\pi}{2} \text{ or, } \sqrt{\frac{2mE_n}{\hbar^2}} = \frac{n\pi}{a} \text{ or, } \frac{2mE_n}{\hbar^2} = \frac{n^2\pi^2}{a^2} \text{ or, } E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

This shows that $E_n \propto 1/a^2$ for a particular value of $n$ and $m$. There is thus an infinite sequence of discrete energy levels.

Similar results are obtained if the potential box is given by, $V(x) = \begin{cases} 0, & \text{for} -a < x < a \\ +\infty, & \text{for } x < -a \text{ and } x > a \end{cases}$

## 5.4 Particle in a 1-D symmetric finite potential well:

Consider a particle of mass '$m$' and energy $E\left(< V_0\right)$ is moving along x-axis, in the region from $x = 0$ to $x = a$ under the following potential i.e.

$$\begin{aligned} V(x) &= 0 \qquad -\frac{a}{2} < x < \frac{a}{2} \\ &= V_0 \qquad \text{otherwise} \end{aligned}$$

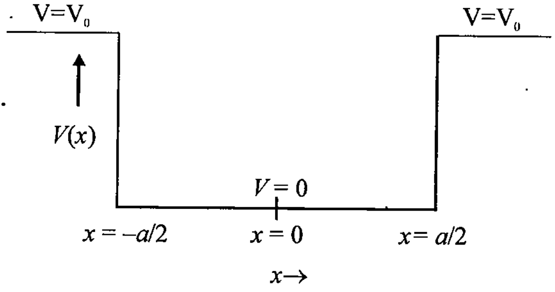


**Figure :** Schematic diagram of 1-D finite potential well.

The time independent 1-D schrodinger equation in three regions can be written as

**Region-1:** $-\frac{\hbar^2}{2m}\frac{d^2\psi_1}{dx^2} + V_0\psi_1 = E\psi_1$ (5a)

**Region-2:** $-\frac{\hbar^2}{2m}\frac{d^2\psi_2}{dx^2} = E\psi_2$ (5b)

**Region-3:** $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi_3 = E\psi_3$ (5c)

Aasuming $\alpha^2 = \frac{2m\left(V_0 - E\right)}{\hbar^2}$ and $\beta^2 = \frac{2mE}{\hbar^2}$, equations (5a), (5b) and (5c) can be written as

**Region-1:** $\frac{d^2\psi_1}{dx^2}-\alpha^2\psi_1=0$ (6a)

**Region-2:** $\frac{d^2\psi_2}{dx^2}+\beta^2\psi_2=0$ (6b)

**Region-3:** $\frac{d^2\psi_3}{dx^2}-\alpha^2\psi_3=0$ (6c)

Solution of equations (6a), (6b) and (6c) will be of the following form:

$$\psi_1(x)=Ae^{\alpha x}+Be^{-\alpha x} \qquad \left(x\le -\frac{a}{2}\right)$$

$$\psi_2(x)=C\cos\beta x+D\sin\beta x \qquad \left(-\frac{a}{2}<x<\frac{a}{2}\right)$$

$$\psi_3(x)=F\,e^{-\alpha x}+G\,e^{\alpha x} \qquad \left(x\ge\frac{a}{2}\right)$$

**Boundary conditions:**

(i) $\psi(x)\to 0$ at $x\to\pm\infty$ (ii) $\psi(x)$ and $\frac{d\psi}{dx}$ are continuous at $x=\pm\frac{a}{2}$

Using the first condition, we get B = 0 and G = 0.
Since the particle is moving under a symmetric potential, then the wavefunction of the particle should have a definite parity i.e. either even or odd.

**(1) Even bound states:**

The wave function of the particle will be of the form:

$$\psi_1(x)=Ae^{\alpha x} \qquad \left(x\le -\frac{a}{2}\right)$$

$$\psi_2(x)=C\cos\beta x \qquad \left(-\frac{a}{2}<x<\frac{a}{2}\right)$$

$$\psi_3(x)=F\,e^{-\alpha x} \qquad \left(x\ge\frac{a}{2}\right)$$

Now, boundary conditions are

$$\psi_1\left(x=-\frac{a}{2}\right)=\psi_2\left(x=-\frac{a}{2}\right)\Rightarrow Ae^{-\alpha\frac{a}{2}}=C\cos\beta\frac{a}{2} \qquad (9)$$

$$\frac{d\psi_1}{dx}\left(x=-\frac{a}{2}\right)=\frac{d\psi_2}{dx}\left(x=-\frac{a}{2}\right)\Rightarrow \alpha Ae^{-\alpha\frac{a}{2}}=\beta C\sin\beta\frac{a}{2} \qquad (10)$$

Dividing equation (10) by (9), we get, $\beta\tan\beta\frac{a}{2}=\alpha$ (11)

We know that, $\alpha^2+\beta^2=\frac{2mV_0}{\hbar^2}$ (12)

Defining $\beta\frac{a}{2}=p$ and $\alpha\frac{a}{2}=q$, we get

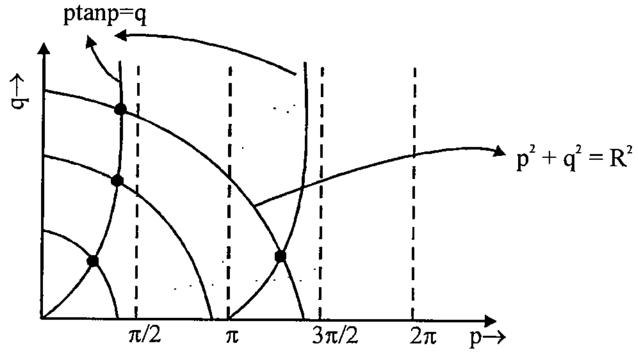
$$p\tan p=q \text{ and } p^2+q^2=\frac{mV_0a^2}{2\hbar^2} \qquad (13)$$

The second equation represents a circle in the p-q plane with radius $R=\sqrt{\frac{mV_0a^2}{2\hbar^2}}$ which depends upon the the height $V_0$ and width $a$ of the finite potential well. Both of these can be solved graphically and the points of intersection of the curves $p\tan p = q$ and $p^2+q^2=\frac{mV_0a^2}{2\hbar^2}$ gives different $p$ values i.e.

$$p_n = p_0, p_2, p_4, \ldots\ldots\ldots$$

which also gives the discrete spectrum of energy values for the even eigenstates as

$$E_n=\frac{2\hbar^2p^2}{ma^2}\quad (n=0,2,4,6\ldots\ldots\ldots)$$



Number of even bound states:

$0<R<\pi \Rightarrow$ one even bound state

$\pi<R<2\pi \Rightarrow$ two even bound state

In general, $n\pi<R<(n+1)\pi \Rightarrow (n+1)$ even bound state

**(2) Odd bound states:**

The wave function of the particle will be of the form:

$$\psi_1(x)=Ae^{\alpha x}\qquad \left(x\le -\frac{a}{2}\right)$$

$$\psi_2(x)=D\sin\beta x\qquad \left(-\frac{a}{2}<x<\frac{a}{2}\right)$$

$$\psi_3(x)=F\,e^{-\alpha x}\qquad \left(x\ge \frac{a}{2}\right)$$

Now, boundary conditions are

$$\psi_1\left(x=-\frac{a}{2}\right)=\psi_2\left(x=-\frac{a}{2}\right)\Rightarrow Ae^{-\alpha\frac{a}{2}}=-D\sin\beta\frac{a}{2} \tag{15}$$

$$\frac{d\psi_1}{dx}\left(x=-\frac{a}{2}\right)=\frac{d\psi_2}{dx}\left(x=-\frac{a}{2}\right)\Rightarrow \alpha Ae^{-\alpha\frac{a}{2}}=\beta D\cos\beta\frac{a}{2} \tag{16}$$

Dividing equation (10) by (9), we get, $\beta\cot\beta\frac{a}{2}=-\alpha$ (17)

We know that, $\alpha^2+\beta^2=\frac{2mV_0}{\hbar^2}$ (18)

Defining $\beta\frac{a}{2}=p$ and $\alpha\frac{a}{2}=q$, we get

$$p\cot p = -q \text{ and } p^2 + q^2 = \frac{mV_0a^2}{2\hbar^2} \tag{19}$$

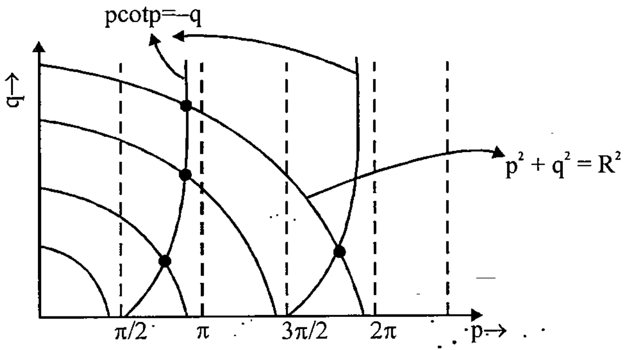
Both of these can be solved graphically and the points of intersection of the curves

$p\cot p = -q$ and $p^2 + q^2 = \dfrac{mV_0a^2}{2\hbar^2}$ gives different $p$ values i.e.

$$p_n = p_1, p_3, p_5, \ldots\ldots\ldots$$

which also gives the discrete spectrum of energy values for the even eigenstates as

$$E_n = \frac{2\hbar^2p^2}{ma^2} \quad (n = 1,3,5,7\ldots\ldots\ldots)$$



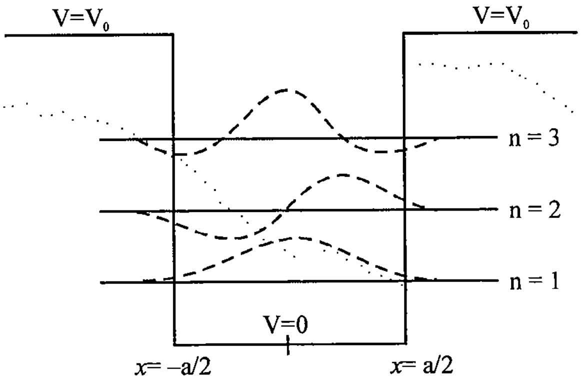
Number of Odd bound states:

$\dfrac{\pi}{2} < R < \dfrac{3\pi}{2} \Rightarrow$ one odd bound state

$\dfrac{3\pi}{2} < R < \dfrac{5\pi}{2} \Rightarrow$ two odd bound state

In general, $(2n-1)\dfrac{\pi}{2} < R < (2n+1)\dfrac{\pi}{2} \Rightarrow n$ odd bound state

| Range of R | No. of bound states | Even | Odd |
|---|---|---|---|
| $0 < R < \frac{\pi}{2}$ | 1 | 1 | 0 |
| $\frac{\pi}{2} < R < \pi$ | 2 | 1 | 1 |
| $\pi < R < \frac{3\pi}{2}$ | 3 | 2 | 1 |
| $\frac{3\pi}{2} < R < 2\pi$ | 4 | 2 | 2 |



**Example 12:** There are only 3 bound states for a particle of mass 'm' in a 1-D potential well of the form:

$$V(x) = 0 \qquad -\frac{a}{2} < x < \frac{a}{2}$$
$$= V_0 \qquad \text{otherwise}$$

The depth of the potential well satisfies the condition.

(a) $\frac{2\pi^2\hbar^2}{ma^2} < V_0 < \frac{9\pi^2\hbar^2}{2ma^2}$ (b) $\frac{2\pi^2\hbar^2}{2ma^2} < V_0 < \frac{4\pi^2\hbar^2}{2ma^2}$

(c) $\frac{2\pi^2\hbar^2}{2ma^2} < V_0 < \frac{8\pi^2\hbar^2}{ma^2}$ (d) $\frac{2\pi^2\hbar^2}{2ma^2} < V_0 < \frac{50\pi^2\hbar^2}{ma^2}$

**Soln.** For three bound states of the particle, $\pi < R < \frac{3\pi}{2}$

$$\Rightarrow \pi^2 < R^2 < \frac{9\pi^2}{4} \Rightarrow \pi^2 < \frac{mV_0a^2}{2\hbar^2} < \frac{9\pi^2}{4} \qquad \Rightarrow \frac{2\pi^2\hbar^2}{ma^2} < V_0 < \frac{9\pi^2\hbar^2}{2ma^2}$$

**Example 13.** Consider a particle of mass 'm' is confined to a finite potential will

$$V(x) = 0 \qquad \text{if } |x| < L$$
$$= V_0 \qquad \text{if } |x| \geq L$$

The particle is found to be in a bound state with energy $\frac{\pi^2\hbar^2}{32ma^2}$. The depth of the well is equal to

(a) $\frac{\pi^2\hbar^2}{2mL^2}$ (b) $\frac{\pi^2\hbar^2}{4mL^2}$ (c) $\frac{\pi^2\hbar^2}{8mL^2}$ (d) $\frac{\pi^2\hbar^2}{16mL^2}$

**Soln.** $\beta^2 = \frac{2mE}{\hbar^2} = \frac{\pi^2}{16L^2} \qquad \Rightarrow \beta = \frac{\pi}{4L}$

Since, the bound state is the ground state of the particle, therefore

$$\alpha = \beta\tan(\beta L) \Rightarrow \beta = \frac{\pi}{4L}\tan\left(\frac{\pi}{4}\right) = \frac{\pi}{4L}$$

Now, $\alpha^2 + \beta^2 = \frac{2mV_0}{\hbar^2} \qquad \Rightarrow V_0 = 2\frac{\pi^2}{16L^2} \times \frac{\hbar^2}{2m} = \frac{\pi^2\hbar^2}{16mL^2}$

**Correct answer is (d)**

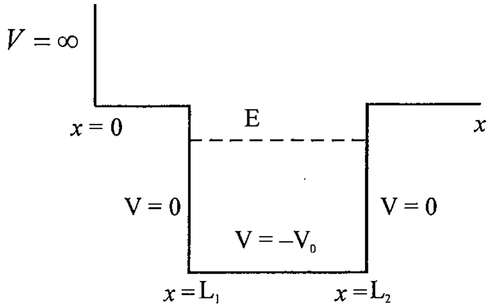
**Example 14.** Show that the ground state energy of a particle of mass 'm' in a finite square well potential of width L is less than that in infinite square well potential of the same width.

**Soln.** The ground state energy eigenvalue can be obtained from the intersection of the first branch of the curve $p\tan p = q$ and $p^2 + q^2 = \frac{mV_0L^2}{2\hbar^2}$. This point of intersection must corresponds to a value of $p$ less than $\pi/2$.

This becuase at $p = \frac{\pi}{2}$, the function $q = p\tan p$ goes to infinity and for $p > \frac{\pi}{2}$, the first branch does not exists.

Therefore, $\frac{2mE}{\hbar^2}\frac{L^2}{4} < \frac{\pi^2}{4} \Rightarrow E < \frac{\pi^2\hbar^2}{2mL^2}$

**Example 15.** Consider a particle in a potential shown in figure below.



Write down the wave functions of the particle (having energy eigenvalue E) in the different regions for bound states.

**Soln.** As is clear from the figure, the energy E should be negative for bound states. We can define four regions.

**Region-I:** $x<0$, **Region-II**: $0<x<L_1$, **Region-III:** $L_1<x<L_2$ and **Region-IV:** $x>L_2$. The wave function for a bound state at energy E can be written as, (solving 1-D time independent Schrodinger equation)

$$\phi_I(x)=0;\ \phi_{II}(x)=Ae^{\alpha x}+Be^{-\alpha x}$$

$$\phi_{III}(x)=C\sin\beta x+D\cos\beta x$$

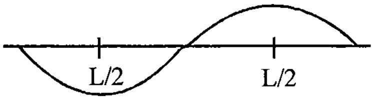
$$\phi_{IV}(x)=F\,e^{-\alpha x}$$

where, $\alpha=\sqrt{\dfrac{-2mE}{\hbar^2}}$ and $\beta=\sqrt{\dfrac{2m(V_0+E)}{\hbar^2}}$

**Example 16.** A particle is in its first excited state in a finite square well potential extended from $x=-\dfrac{L}{2}$ to $x=\dfrac{L}{2}$. The maximum value of the wave function, at a particular instant, is $\psi=A$ and it occurs at $x=\dfrac{L}{3}$. Find the value of the function at $x=\dfrac{L}{2}$.

**Soln.** In the first excited state, the wave function is of the form $\psi=A\sin kx$. The maximum value of the wave function is A. According to the question, this occurs at $x=\dfrac{L}{3}$. This means,

$$A=A\sin\frac{kL}{3}\quad\Rightarrow\frac{kL}{3}=\frac{\pi}{2}\quad\Rightarrow k=\frac{3\pi}{2L}$$



The wave function at $x=\dfrac{L}{2}$ is $\psi=A\sin\dfrac{kL}{2}=A\sin\left(\dfrac{3\pi}{2L}\times\dfrac{L}{2}\right)=\dfrac{A}{\sqrt{2}}$

**Example 17.** A finite square well potential has length L and depth $V_0$. A particle of mass $\dfrac{2\hbar^2}{V_0L^2}$ is trapped in this potential. Show that it has only one bound state and the energy corresponding to this states satisfies $\cos\left(\sqrt{E/V_0}\right)=\sqrt{E/V_0}$

**Soln.** $p^2+q^2=\left(\frac{L}{2}\right)^2\left[\frac{2mV_0}{\hbar^2}\right]=\frac{V_0L^2}{2\hbar^2}\times\frac{2\hbar^2}{V_0L^2}=1$

Thus, $\sqrt{p^2+q^2}<\frac{\pi}{2}$ and hence only one bound state occurs.

For the energy E, $q=p\tan p\Rightarrow\cos p=\frac{1}{\sqrt{1+\tan^2 p}}=\frac{p}{\sqrt{p^2+q^2}}=p$

Using, $p=\beta\frac{L}{2}=\frac{L}{2}\times\sqrt{\frac{2mE}{\hbar^2}}=\frac{L}{2}\sqrt{\frac{2E}{\hbar^2}\times\frac{2\hbar^2}{V_0L^2}}=\sqrt{\frac{E}{V_0}}$

Therefore, $\cos\left(\sqrt{\frac{E}{V_0}}\right)=\sqrt{\frac{E}{V_0}}$

## 5.5. Particle in 3-D rectangular box :

We consider a free particle of mass m confined to move non-relativistically in a three-dimensional potential box of sides $a$, $b$ and $c$ parallel to the $x, y$ and $z$-axes respectively. Since the particle is free, the potential inside the box is $V(x,y,z)=0$. So, the total energy of the particle inside the box is equal to its kinetic energy which is always a positive quantity.

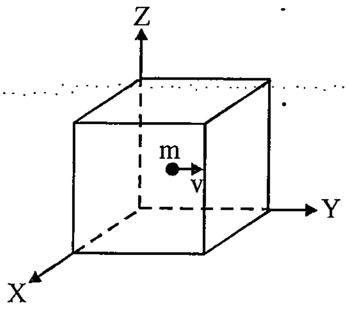


**Figure :** Schematic diagram of 3-D infinite potential well or 3-D box

If v be the velocity, then the momentum of the particle $p = mv$ and its energy

$$E=\frac{1}{2}mv^2=\frac{p^2}{2m}$$

Since the potential energy $V(x,y,z)$ is independent of time, we here apply Schrodinger time -independent 3-D wave equation to obtain the wave functiuon $\psi(x,y,z)$ and the eigenvalue of the energy of the particle. The three-dimensional potential box has the following characteristics:

$$V(x,y,z)=\begin{cases}0, & \text{for } 0<x<a,\ 0<y<b,\ 0<z<c\\ \infty & \text{for}\quad x>a,\quad y>b,\quad z>c\end{cases}\qquad(1)$$

**The wave equation for the particle:**

The wave equation describing the motion of the particle can be written as

$$\hat{H}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi = E\psi \Rightarrow \frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+\frac{2mE}{\hbar^2}\psi = 0 \tag{2}$$

As $V(x,y,z)=\infty$ at the boundries and outside the box $\psi(x,y,z)=0$ at the boundaries and outside the potential box. The boundary conditions can be written as,

$$\left.\begin{aligned}\psi(x,y,z)=0 \text{ at } x=0 \text{ and } x=a\\ y=0 \text{ and } y=b\\ z=0 \text{ and } z=c\end{aligned}\right\}$$

Using the separation of variable technique, we caan assume the solution of equation (2) to be

$$\psi(x,y,z)=X(x)Y(y)Z(z) \tag{3}$$

Substituting equation (3) in equation (2), we get

$$YZ\frac{d^2X}{dx^2}+ZX\frac{d^2Y}{dy^2}+XY\frac{d^2Z}{dz^2}+\frac{2mE}{\hbar^2}XYZ=0$$

Dividing throughout by XYZ, we get

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}+\frac{2mE}{\hbar^2}=0 \tag{4}$$

If $v_x, y_y$ and $v_z$ be the component of velocity of the particle along the $x$, $y$ and $z$-axes respectively, then the corresponding kinetic energies of the particle are

$$E_x=\frac{1}{2}mv_x^2,\quad E_y=\frac{1}{2}mv_y^2 \text{ and } E_z=\frac{1}{2}mv_z^2 \text{ such that } E=E_x+E_y+E_z$$

Therefore, equation (4) can be written as

$$\left[\frac{1}{X}\frac{d^2X}{dx^2}+\frac{2mE_x}{\hbar^2}\right]+\left[\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{2mE_y}{\hbar^2}\right]+\left[\frac{1}{Z}\frac{d^2Z}{dz^2}+\frac{2mE_z}{\hbar^2}\right]=0 \tag{5}$$

So,

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{2mE_x}{\hbar^2}X=0 \tag{6a}$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{2mE_y}{\hbar^2}Y=0 \tag{6b}$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2}+\frac{2mE_z}{\hbar^2}Z=0 \tag{6c}$$

Solutions of equations (6a), (6b) and (6c) are

$$X(x)=A_1\sin k_xx+B_1\cos k_xx \quad \text{where } k_x=\frac{\sqrt{2mE_x}}{\hbar}$$

$$Y(y)=A_2\sin k_yy+B_2\cos k_yy \quad \text{where } k_y=\frac{\sqrt{2mE_y}}{\hbar}$$

$$Z(z)=A_3\sin k_zz+B_3\cos k_zz \quad \text{where } k_z=\frac{\sqrt{2mE_z}}{\hbar}$$

**Boundary Conditions:**

(i) $X(x=0)=0 \Rightarrow B_1=0$

$$X(x=a)=0 \Rightarrow A\sin k_x a=0 \quad \Rightarrow \sin k_x a=0 \quad \Rightarrow k_x=\frac{n_x\pi}{a} \quad (n_x=1,2,3.......)$$

(ii) $Y(y=0)=0 \Rightarrow B_2=0$

$$Y(y=b)=0 \Rightarrow A_2\sin k_y a=0 \quad \Rightarrow \sin k_y a=0 \quad \Rightarrow k_y=\frac{n_y\pi}{a} \quad (n_y=1,2,3.......)$$

(iii) $Z(z=0)=0 \Rightarrow B_3=0$

$$Z(z=c)=0 \Rightarrow A_3\sin k_z a=0 \quad \Rightarrow \sin k_z a=0 \quad \Rightarrow k_z=\frac{n_z\pi}{a} \quad (n_z=1,2,3.......)$$

Therefore, the solution will become

$$\psi(x,y,z)=A_1A_2A_3\sin\frac{n_x\pi}{a}x\sin\frac{n_y\pi}{b}y\sin\frac{n_z\pi}{c}z$$

$$\Rightarrow \psi(x,y,z)=A\sin\frac{n_x\pi}{a}x\sin\frac{n_y\pi}{b}y\sin\frac{n_z\pi}{c}z \tag{7}$$

**Normalization Condition:**

$$\int_0^a\int_0^b\int_0^c |\psi|^2\,dxdydz=1$$

$$\Rightarrow A^2\int_0^a\int_0^b\int_0^c \sin^2\frac{n_x\pi}{a}x\sin^2\frac{n_y\pi}{b}y\sin^2\frac{n_z\pi}{c}z\,dxdydz=1$$

$$\Rightarrow A^2.\frac{a}{2}.\frac{b}{2}.\frac{c}{2}=1 \Rightarrow A=\sqrt{\frac{8}{abc}}$$

Now, the eigenfunction of the particle is,

$$\psi(x,y,z)=\sqrt{\frac{8}{abc}}\sin\frac{n_x\pi}{a}x\sin\frac{n_y\pi}{b}y\sin\frac{n_z\pi}{c}z \tag{8}$$

Therefore, the total linear momentum $p$ is $p=p_x^2+p_y^2+p_z^2$

$$=\left[\left(\frac{n_x\hbar\pi}{a}\right)+\left(\frac{n_y\hbar\pi}{b}\right)^2+\left(\frac{n_z\hbar\pi}{c}\right)^2\right]^{1/2}=\pi\hbar\left[\left(\frac{n_x}{a}\right)^2+\left(\frac{n_y}{b}\right)^2+\left(\frac{n_z}{c}\right)^2\right]$$

The eigenvalue of energy of the particle is,

$$E=\frac{p^2}{2m}=\frac{\pi^2\hbar^2}{2m}\left[\left(\frac{n_x}{a}\right)^2+\left(\frac{n_y}{b}\right)^2+\left(\frac{n_z}{c}\right)^2\right]$$

The integers $n_x, n_y, n_z$ are called quantum numbers which are required to describe the stationary state of the particle. Since for any of $n_x, n_y, n_z$ being zero, $\psi(x,y,z)=0$, i.e., the particle is absent in the box. But physically this is not acceptable and hence any of $n_x, n_y, n_z$ cannot be zero.

For the ground state, $n_x = n_y = n_z = 1$ and the ground state wave function of the particle is,

$$\psi_{1,1,1}(x,y,z) = \sqrt{\frac{8}{abc}} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \sin\frac{\pi z}{c}$$

and the energy associated with the ground state called the zero point energy is given by,

$$E_{1,1,1} = \frac{\pi^2\hbar^2}{2m}\left[\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right].$$

The eigenfunctions and eigenvalues of energy for the next higher excited state are as follows:

$$\left.\begin{array}{l} \psi_{2,1,1} = \sqrt{\dfrac{8}{abc}} \sin\dfrac{2\pi x}{a} \sin\dfrac{\pi y}{b} \sin\dfrac{\pi z}{c} \\ \psi_{1,2,1} = \sqrt{\dfrac{8}{abc}} \sin\dfrac{\pi x}{a} \sin\dfrac{2\pi y}{b} \sin\dfrac{\pi z}{c} \\ \psi_{1,1,2} = \sqrt{\dfrac{8}{abc}} \sin\dfrac{\pi x}{a} \sin\dfrac{\pi y}{b} \sin\dfrac{2\pi z}{c} \end{array}\right\} \text{ and } \left.\begin{array}{l} E_{2,1,1} = \dfrac{\pi^2\hbar^2}{2m}\left[\dfrac{4}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}\right] \\ E_{1,2,1} = \dfrac{\pi^2\hbar^2}{2m}\left[\dfrac{1}{a^2}+\dfrac{4}{b^2}+\dfrac{1}{c^2}\right] \\ E_{1,1,2} = \dfrac{\pi^2\hbar^2}{2m}\left[\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{4}{c^2}\right] \end{array}\right\}$$

## 5.6. Particle in 3-D Cubical box:

If the potential box be cubical instead of rectagular, then $a = b = c$ the eigenfunction corresponding to the state of the particle will be represented by

$$\psi_{n_x n_y n_z}(x,y,z) = \sqrt{\frac{8}{a^3}} \sin\frac{n_x\pi}{a}x \sin\frac{n_y\pi}{a}y \sin\frac{n_z\pi}{a}z$$

The momentum eigen values of the particle will be,

$$p_{n_x,n_y,n_z} = \frac{\pi\hbar}{a}\left[n_x^2+n_y^2+n_z^2\right]^{1/2}$$

and the corresponding energy eigenvalues are, $E_{n_x,n_y,n_z} = \dfrac{\pi^2\hbar^2}{2ma^2}\left[n_x^2+n_y^2+n_z^2\right]$

Ground state ($n_x = n_y = n_z = 1$):

**Wave function:** $\psi_{1,1,1} = \sqrt{\dfrac{8}{a^3}} \sin\dfrac{\pi x}{a} \sin\dfrac{\pi y}{a} \sin\dfrac{\pi z}{a}$

**Momentum :** $p_{1,1,1} = \dfrac{\pi\hbar}{a}\sqrt{3}$ **Energy :** $E_{1,1,1} = \dfrac{3\pi^2\hbar^2}{2ma^2}$

**First excited state:**

| $n_x$ | $n_y$ | $n_z$ |
|---|---|---|
| 2 | 1 | 1 |
| 1 | 2 | 1 |
| 1 | 1 | 2 |

Wave function:
$$\left.\begin{array}{l} \psi_{2,1,1} = \sqrt{\dfrac{8}{a^3}} \sin\dfrac{2\pi x}{a} \sin\dfrac{\pi y}{a} \sin\dfrac{\pi z}{a} \\ \psi_{1,2,1} = \sqrt{\dfrac{8}{a^3}} \sin\dfrac{\pi x}{a} \sin\dfrac{2\pi y}{a} \sin\dfrac{\pi z}{a} \\ \psi_{1,1,2} = \sqrt{\dfrac{8}{a^3}} \sin\dfrac{\pi x}{a} \sin\dfrac{\pi y}{a} \sin\dfrac{2\pi z}{a} \end{array}\right\}$$

Momentum: $p_{2,1,1} = p_{1,2,1} = p_{1,1,2} = \frac{\pi\hbar}{a}\sqrt{6}$

Energy: $E_{2,1,1} = E_{1,2,1} = E_{1,1,2} = \frac{\pi^2\hbar^2}{2ma^2}6$

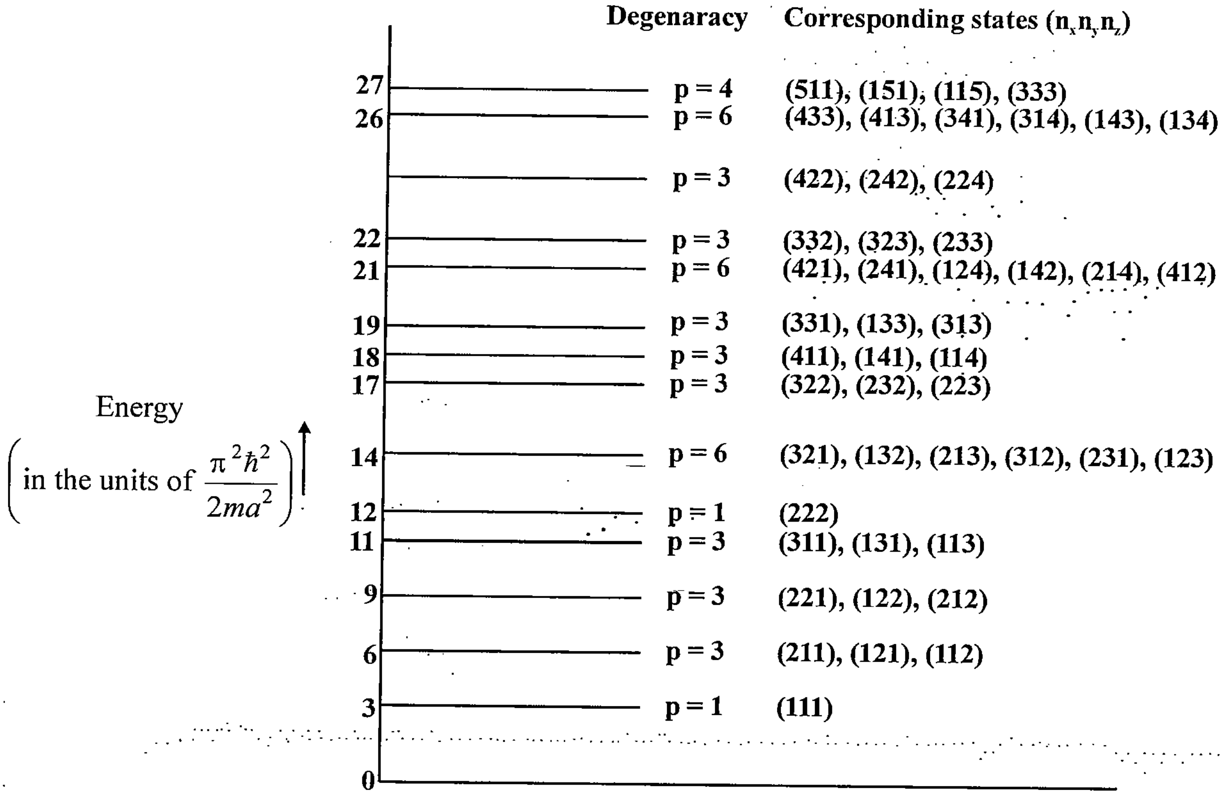

Degenaracy
Corresponding states ($n_x n_y n_z$)
27 p = 4 (511), (151), (115), (333)
26 p = 6 (433), (413), (341), (314), (143), (134)
p = 3 (422), (242), (224)
22 p = 3 (332), (323), (233)
21 p = 6 (421), (241), (124), (142), (214), (412)
19 p = 3 (331), (133), (313)
18 p = 3 (411), (141), (114)
17 p = 3 (322), (232), (223)
14 p = 6 (321), (132), (213), (312), (231), (123)
12 p = 1 (222)
11 p = 3 (311), (131), (113)
9 p = 3 (221), (122), (212)
6 p = 3 (211), (121), (112)
3 p = 1 (111)
0
Energy
(in the units of $\frac{\pi^2\hbar^2}{2ma^2}$)

**Figure :** Energy levels of a particle in a cubical potential box

**Degeneracy:**

The property that, two or more quantum states of a particle, with different sets of quantum numbers and different eigenfunctions having the same value of momentum and energy is called degeneracy.

For example, the three states corresponding to quantum numbers (2, 1, 1), (1, 2, 1) and (1, 1, 2) all have the same momentum $\left(\frac{\pi\hbar}{a}\sqrt{6}\right)$ and energy $\left(\frac{\pi^2\hbar^2}{2ma^2}6\right)$.

**Example 18.** An electron is confined in a cubical box of side 1Å. Find the momentum and energy of the electron in the ground state and the first excited state.

**oln.** Ground state: $n_x = n_y = n_z = 1$

Momentum of the electron is

$$P_{1,1,1} = \frac{\pi\hbar}{\ell}\sqrt{3} = \frac{\pi h}{2\pi\ell}\sqrt{3} = \frac{h}{2\ell}\sqrt{3} = 5.72\times10^{-24} kg - m/s$$

Energy of the electron is

$$E_{1,1,1} = \frac{3\pi^2\hbar^2}{2m\ell^2} = 181\times10^{-19} J = 113\, eV$$

**First excited state:**

| $n_x$ | $n_y$ | $n_z$ |
|---|---|---|
| 2 | 1 | 1 |
| 1 | 2 | 1 |
| 1 | 1 | 2 |

Momentum of the electron: $p_{2,1,1}=p_{1,2,1}=p_{1,1,2}=\frac{\pi\hbar}{a}\sqrt{6}=8.13\times10^{-24}\,kg-m/s$

Energy of the electron : $E_{2,1,1}=E_{1,2,1}=E_{1,1,2}=\frac{\pi^2\hbar^2}{2ma^2}6=362\times10^{-19}\,J=226\,eV$

**Example 19.** An electron is confined in a cubical box of side 1Å. (i) Find its lowest energy (ii) what is the temperature at which the average of the molecule of a perfect gas would be equal to the lowest energy of the electron $\left(k=1.38\times10^{-23}\,J/K\right)$?

**Soln.** (i) Ground state energy of the electron

$$E_{1,1,1}=\frac{3\pi^2\hbar^2}{2ma^2}=\frac{3\pi^2\times\left(1.054\times10^{-34}\right)^2}{2\times9.11\times10^{-31}\times\left(10^{-10}\right)^2}=18.03\times10^{-18}\,J$$

(ii) If T K be the required temeprature of the molecules, then

$$\frac{3}{2}kT=\frac{3\pi^2\hbar^2}{2ma^2}\quad\text{[k is the Boltzmann constant]}$$

$$=18.03\times10^{-18}\,J$$

$$\therefore\quad T=\frac{2\times18.03\times10^{-18}}{3\times1.38\times10^{-23}}=8.71\times10^{5}\,K$$

**Example 20.** A particle of mass 'm' is in a cubic box of size 'a'. If the particle is in an eigenstate with energy $E=\frac{14\pi^2\hbar^2}{2ma^2}$. What will be its wave function?

**Soln.** $E=\frac{14\pi^2\hbar^2}{2ma^2}=\frac{\pi^2\hbar^2}{2ma^2}\left(n_x^2+n_y^2+n_z^2\right)\Rightarrow n_x^2+n_y^2+n_z^2=14$

$$\left(n_x,n_y,n_z\right)=(1,2,3)$$

$$\psi_{123}(x,y,z)=\sqrt{\frac{8}{a^3}}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right)\sin\left(\frac{3\pi z}{a}\right)$$

$$=\left(\frac{2}{a}\right)^{3/2}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right)\sin\left(\frac{3\pi z}{a}\right)$$

**Example 21.** Find the number of energy states and energy levels in the range $E<[15h^2/(8\text{ma}^2)]$ of a cubical box of side $a$.

**Soln.** For a particle in a cubic box of side $a$, the energy is given by

$$E=\frac{\pi^2\hbar^2}{2ma^2}(n_x^2+n_y^2+n_z^2)=\frac{h^2}{8ma^2}(n_x^2+n_y^2+n_z^2)$$

Comparing with the given expression, we get $n_x^2 + n_y^2 + n_z^2 < 15$

The number of possible combinations of $(n_x\, n_y\, n_z)$ is

| | |
|---|---|
| (1 1 1) | 1 way |
| (1 1 2), (1 2 1), (2 1 1) | 3 ways |
| (1 1 3), (1 3 1), (3 1 1) | 3 ways |
| (1 2 2), (2 1 2), (2 2 1) | 3 ways |
| (2 2 2) | 1 way |
| (1 2 3), (1 3 2), (2 1 3), (3 2 1), (3 1 2) | 6 ways |
| Total | 17 ways |

Hence the No. of possible states = 17. The No. of energy levels = 6.

**Example 22.** A crystal has some negative ion vacancies, each containing one electron. Treat these electrons as moving freely inside a volume whose dimensions are of the order of lattice constant. Assuming the value of lattice constant, estimate the longest wavelength of electromagnetic radiation absorbed by these electrons.

**Soln.** The energy levels of an electron in a cubical box of side $a$ is

$$E_{n_x,n_y,n_z} = \frac{\pi^2\hbar^2}{2ma^2}(n_x^2 + n_y^2 + n_z^2), \quad n_x, n_y, n_z = 1, 2, 3, \ldots$$

Lattice constant a = 1 Å = $10^{-10}$ m.

The energy of the ground state is given by $E_{1111} = \frac{\pi^2\hbar^2}{2ma^2} \times 3 = 1.795 \times 10^{-17} J$

The longest wavelength corresponds to the transition from energy $E_{111}$ to $E_{211}$, and hence

$$E_{211} = \frac{\pi^2\hbar^2 \times 6}{2ma^2} = 3.59 \times 10^{-17}\,\mathrm{J}$$

Longest wavelength $\lambda = \frac{c}{v} = \frac{ch}{E_{211} - E_{111}} = 11\, nm$

## 5.7. Particle in 3-D Harmonic Oscillator :

We consider a particle of mass $m$ bound to the origin by a force having components.

$$F_x = -k_x x = -m\omega_x^2 x \;\; ; \;\; F_y = -k_y x = -m\omega_y^2 y \;\; ; \;\; F_z = -k_z z = -m\omega_z^2 z$$

along the $x, y, z$ components of the displacement $\vec{r}$ from the origin. Here, $k_x, k_y, k_z$ are the force constants and $\omega_x, \omega_y, \omega_z$ are the angular frequencies along $x, y, z$-axes respectively. Such a system is called a three-dimensional harmonic oscillator.

Again, $$F_x = -\frac{\partial V}{\partial x}, \quad F_y = -\frac{\partial V}{\partial y} \text{ and } F_z = -\frac{\partial V}{\partial z}$$

where the potential energy $V$ of the oscillator is,

$$V(x, y, z) = \frac{1}{2}\left(k_x x^2 + k_y y^2 + k_z z^2\right)$$

The time-independent three-dimensional Schrodinger equation for the system is

$$-\frac{\hbar^2}{2m}\nabla^2\psi(x,y,z)+\frac{1}{2}\left(k_x x^2+k_y y^2+k_z z^2\right)\psi(x,y,z)=E\psi(x,y,z)$$

$$\Rightarrow \frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+\frac{2m}{\hbar^2}\left[E-\frac{1}{2}k_x x^2-\frac{1}{2}k_y y^2-\frac{1}{2}k_z z^2\right]=0$$

$$\Rightarrow \frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+\left[\lambda-\left(\alpha_x^2x^2+\alpha_y^2y^2+\alpha_z^2z^2\right)\right]\psi=0 \tag{9}$$

where, $\lambda=\frac{2mE}{\hbar^2}, \alpha_x=\sqrt{\frac{mk_x}{\hbar^2}}, \alpha_y=\sqrt{\frac{mk_y}{\hbar^2}}$ and $\alpha_z=\sqrt{\frac{mk_y}{\hbar^2}}$

Equation (9) can be solved by the method of separation of variables. The energy eigenvalues and eigenfunctions of the three-dimensional oscillator are therefore,

$$\left.\begin{aligned} E_x&=\left(n_x+\frac{1}{2}\right)\hbar\omega_x\\ E_y&=\left(n_y+\frac{1}{2}\right)\hbar\omega_y\\ E_z&=\left(n_z+\frac{1}{2}\right)\hbar\omega_z \end{aligned}\right\}$$

where, $n_x, n_y$ and $n_z$ are positive integers i.e., $n_x, n_y, n_z = 1, 2, 3, \ldots\ldots$ and

$$X(x)=N_{n_x}\,e^{-\alpha_x^2x^2/2}H_{n_x}(\alpha_x x);\ Y(y)=N_{n_y}\,e^{-\alpha_y^2y^2/2}H_{n_y}(\alpha_y y);\ Z(z)=N_{n_z}\,e^{-\alpha_z^2z^2/2}H_{n_z}(\alpha_z z)$$

Therefore, the complete wave function is

$$\psi_{n_xn_yn_z}=N_{n_x}N_{n_y}N_{n_z}\,e^{-\left(\alpha_x^2x^2+\alpha_y^2y^2+\alpha_z^2z^2\right)/2}H_{n_x}(\alpha_x x)H_{n_y}(\alpha_y y)H_{n_y}(\alpha_z z) \tag{10}$$

where the normalization factor is

$$N_{n_x\,n_y\,n_z}=\left[\frac{\alpha_x\alpha_y\alpha_z}{\pi^{3/2}2^{(n_x+n_y+n_z)}n_x!n_y!n_z!}\right]^{1/2}$$

$$E_{n_xn_yn_z}=\hbar\left[\left(n_x+\frac{1}{2}\right)\omega_x+\left(n_y+\frac{1}{2}\right)\omega_y+\left(n_z+\frac{1}{2}\right)\omega_z\right]$$

For the isotropic oscillator, $\omega_x=\omega_y=\omega_z$ and $\alpha_x=\alpha_y=\alpha_z$

$$\therefore \quad E_n=\left[n_x+n_y+n_z+\frac{3}{2}\right]\hbar\omega=\left(n+\frac{3}{2}\right)\hbar\omega$$

where $n=n_x+n_y+n_z$ is the total quantum number and n = 0, 1, 2, 3, ........

It can be shown that except for the lowest energy level $\left(n_x=n_y=n_z=0, i.e.\ n=0\right)$ all the energy levels are degenerate.

The zero point (or, the ground level) energy $E_0$ for $n_x=n_y=n_z=0$ of the oscillator is given by

$$E_0=\frac{3}{2}\hbar\omega$$

Thus the zero point energy of a three-dimensional isotropic harmonic oscillator is three times that for the linear harmonic oscillator. the ground level of the oscillator is thus non-degenerate. The state with principal quantum number n = 1, there are three levels each of energy $\frac{5}{2}\hbar\omega$. So, the first excited state is 3-fold degenerate. Similary, the state with n = 2 is 6-fold degenerate. Thus all the energy levels of the isotropic three-dimensional oscillator are degenerate and the degree of degeneracy is $p=\frac{1}{2}(n+1)(n+2)$.

| $n_x$ | $n_y$ | $n_z$ | $n$ | Wave function | $E_n$ | Degeneracy $(p)$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | $\psi_{0,0,0}$ | $\frac{3}{2}\hbar\omega$ | $p=1$ |
| 1 | 0 | 0 | 1 | $\psi_{1,0,0}$ | $\frac{3}{2}\hbar\omega$ | $p=3$ |
| 0 | 1 | 0 | | $\psi_{0,1,0}$ | | |
| 0 | 0 | 1 | | $\psi_{0,0,1}$ | | |
| 1 | 1 | 0 | 2 | $\psi_{1,1,0}$ | $\frac{3}{2}\hbar\omega$ | $p=6$ |
| 1 | 0 | 1 | | $\psi_{1,0,0}$ | | |
| 0 | 1 | 1 | | $\psi_{0,1,1}$ | | |
| 2 | 0 | 0 | 2 | $\psi_{2,0,0}$ | | |
| 0 | 2 | 0 | | $\psi_{0,2,0}$ | | |
| 0 | 0 | 2 | | $\psi_{0,0,2}$ | | |

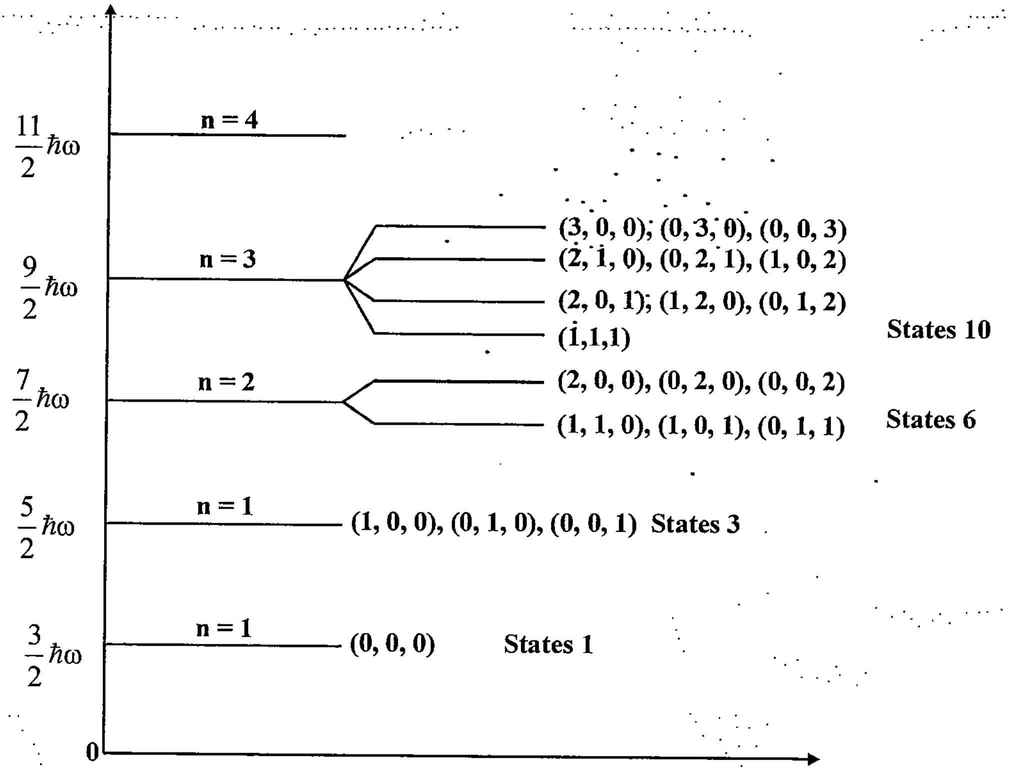

$\frac{11}{2}\hbar\omega$
n = 4
$\frac{9}{2}\hbar\omega$
n = 3
(3, 0, 0), (0, 3, 0), (0, 0, 3)
(2, 1, 0), (0, 2, 1), (1, 0, 2)
(2, 0, 1); (1, 2, 0), (0, 1, 2)
(1,1,1)
States 10
$\frac{7}{2}\hbar\omega$
n = 2
(2, 0, 0), (0, 2, 0), (0, 0, 2)
(1, 1, 0), (1, 0, 1), (0, 1, 1)
States 6
$\frac{5}{2}\hbar\omega$
n = 1
(1, 0, 0), (0, 1, 0), (0, 0, 1) States 3
$\frac{3}{2}\hbar\omega$
n = 1
(0, 0, 0)
States 1
0

**Figure :** Energy levels and degeneracy for the 3-dimensional harmonic oscillator

**Example 23.** The wave function of a particle confined in a box of length $L$ is

$$\psi(x)=\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}$$

The probability of finding the particle in the domain $0<x<\frac{L}{2}$ is **[BHU 2010, 2011]**

(a) 1 (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) 0

**Soln.** Given : $\psi(x)=\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}$

The probability is $P=\int_0^{L/2}|\psi(x)|^2\,dx=\frac{2}{L}\int_0^{L/2}\sin^2\left(\frac{\pi x}{L}\right)dx$

$$=\frac{2}{2L}\int_0^{L/2}\left\{1-\cos\left(\frac{2\pi x}{L}\right)\right\}dx$$

$$=\frac{1}{L}\left[x-\frac{L}{2\pi}\sin\left(\frac{2\pi x}{L}\right)\right]_0^{L/2}=\frac{1}{L}\left[\frac{L}{2}\right]=\frac{1}{2}.$$

**Correct option is (c)**

**Example 24.** The ground state energy of a particle in an one-dimensional quantum well is 4.4 eV. If the width of the well is doubled, then the new ground state energy is **[BHU 2011]**

(a) 1.1 eV (b) 2.2 eV (c) 8.8 eV (d) 17.6 eV

**Soln.** Ground state energy of a particle in a 1-D quantum well, is $E_1=\frac{\pi^2\hbar^2}{2mL^2}$ (where L is the width of the well).

If the width of the well is doubled, then the ground state energy will decrease 4 times.

Therefore, new ground state energy will be 1.1 eV.

**Correct option is (a)**

**Example 25.** If an electron $\left(\mathrm{m}=9.1\times10^{-31}\,\mathrm{kg}\right)$ is confined by an electrical force to move between two rigid walls separated by $1.0\times10^{-9}$ meter. The quantized energy value for the second lowest energy state is

**[BHU 2011]**

(a) 0.38 eV (b) 0.76 eV (c) 1.5 eV (d) 3.0 eV

**Soln.** Second lowest energy = First excited state energy =

$$E_2=\frac{4\pi^2\hbar^2}{2mL^2}=\frac{4\times(3.14)^2\times\left(1.05\times10^{-34}\right)^2}{2\times\left(9.1\times10^{-31}\right)\times\left(10^{-9}\right)^2}=2.38\times10^{-19}J=1.49\,eV$$

**Correct option is (c)**

**Example 26.** A particle is confined in a one-dimensional box of length $l$. Its wave function is given by

$$\psi(x)=\begin{cases}\sqrt{\frac{2}{l}}\sin\left(\frac{\pi x}{l}\right) & \text{for} \quad 0<x<1\\ 0 & \text{for} \quad l<x<0\end{cases}$$

**[BHU 2012]**

The expectation values of the $x$-component of momentum is

(a) 1 (b) 0.5 (c) $\frac{l}{2}$ (d) zero

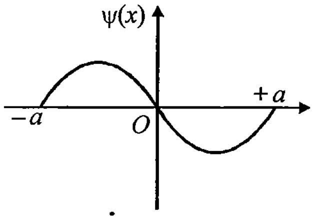
**Soln.** Given : $\psi(x)=\begin{cases}\sqrt{\frac{2}{l}}\sin\left(\frac{\pi x}{l}\right) & \text{for} \quad 0<x<1\\ 0 & \text{for} \quad l<x<0\end{cases}$

$$\therefore \quad \langle \hat{p}_x\rangle=\langle\psi\,|\,\hat{p}\,|\,\psi\rangle=\int_0^l \psi^*(x)\left(-i\hbar\frac{d}{dx}\right)\psi(x)\,dx$$

$$=\frac{2}{l}\int_0^l \frac{\pi}{l}\sin\left(\frac{\pi x}{l}\right)(-i\hbar)\cos\left(\frac{\pi x}{l}\right)dx=-\frac{i\hbar\pi}{l^2}\int_0^l \sin\left(\frac{2\pi x}{l}\right)dx=+\frac{i\hbar\pi}{l^2}\left[\frac{l}{2\pi}\cos\left(\frac{2\pi x}{l}\right)\right]_0^l=0.$$

**Correct option is (d)**

**Example 27.** In the figure below shows one of the possible energy eigenfunctions $\psi(x)$ for a particle bouncing freely back and forth along the $x$-axis between impenetrable walls located at $x=-a$ and $x=+a$. The potential energy equals zero for $|x|<a$. If the energy of the particle is 2 electron volts when it is in the quantum state associated with this eigenfunction, what is its energy when it is in the quantum state of lowest possible energy ?



(a) 0 eV (b) $\frac{1}{\sqrt{2}}$ eV (c) $\frac{1}{2}$ eV (d) 1 eV **[BHU 2012]**

**Soln.** The energy of the particle is given by $E_n=\frac{n^2\pi^2\hbar^2}{2ma^2}$

The given wave function corresponds to $n=2$.

$$\therefore \quad E_2=\frac{4\pi^2\hbar^2}{2ma^2}=2\text{ eV}\Rightarrow\frac{\pi^2\hbar^2}{2ma^2}=\frac{1}{2}\text{ eV}.$$

The lowest possible energy is the corresponds to $n=1$, i.e.,

$$E_1=\frac{\pi^2\hbar^2}{2ma^2}=\frac{1}{2}\text{ eV}.$$

**Correct option is (c)**

**Example 28.** A particle is moving in a one-dimensional potential box of infinite height. What is the probability of finding the particle in a small interval $\Delta x$ at the center of the box when it is in the energy state, next to the least energy state? **[BHU 2012]**

(a) 0 (b) 0.15 (c) 0.35 (d) 0.5

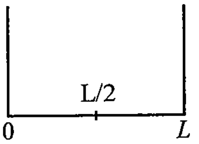
**Soln.** The wave function of the particle is

$$\psi(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$$

The required probability is

$$P=\int_{\frac{L}{2}-\frac{\Delta x}{2}}^{\frac{L}{2}+\frac{\Delta x}{2}}|\psi(x)|^2\,dx$$

$$=\frac{2}{L}\int_{\frac{L}{2}-\frac{\Delta x}{2}}^{\frac{L}{2}+\frac{\Delta x}{2}}\sin^2\left(\frac{2\pi x}{L}\right)dx=\frac{2}{2L}\left[x-\frac{L}{4\pi}\sin\left(\frac{4\pi x}{L}\right)\right]_{\frac{L-\Delta x}{2}}^{\frac{L+\Delta x}{2}}$$



$$=\frac{1}{L}\left[(\Delta x)-\frac{L}{4\pi}\left\{\sin\frac{4\pi}{L}\left(\frac{L+\Delta x}{2}\right)-\sin\frac{4\pi}{L}\left(\frac{L-\Delta x}{2}\right)\right\}\right]$$

$$=\left[\Delta x-\frac{2L}{4\pi}\cos 2\pi\sin\frac{2\pi}{L}\Delta x\right]$$

$$=\frac{1}{L}\left[\Delta x-\frac{L}{2\pi}\sin\frac{2\pi}{L}\Delta x\right]$$

For small $\Delta x$, $\sin\frac{2\pi}{L}\Delta x\approx\frac{2\pi}{L}\Delta x$

Therefore, $P=\frac{1}{L}\left[\Delta x-\frac{L}{2\pi}\frac{2\pi}{L}\Delta x\right]=0$

**Correct option is (a)**

**Example 29.** A particle of mass $m$ is restricted to move in one-dimensional between two points such that $0\le x\le a$. If the potential function is such that **[BHU 2013]**

$$V(x)=\infty \quad x<0 \text{ and } x>a$$
$$=0 \quad 0\le x\le a$$

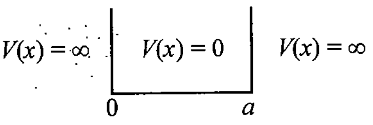
the particle will have discrete energy spectrum as

(a) $E_n=\frac{\pi^2\hbar^2}{4ma^2}\cdot n^2,\ n=0,1,2,\ldots$ (b) $E_n=\frac{3\pi^2\hbar^2}{2ma^2}(n+1)^2,\ n=0,1,2,\ldots$

(c) $E_n=\frac{\pi^2\hbar^2}{2ma^2}(n+1)^2,\ n=0,1,2,\ldots$ (d) $E_n=\frac{3}{2}\frac{\pi^2\hbar^2}{ma^2}\cdot n^2,\ n=0,1,2,\ldots$

**Soln.** This is one-dimensional particle in a box problem.

The energy is given by



$$E_n=(n+1)^2\frac{\pi^2\hbar^2}{2ma^2}\ ;\ n=0,1,2,\ldots.$$

**Correct option is (c)**

**Example 30.** The normalized ground state of a particle of mass $m$ which is constrained to move inside a potential $V(x)=\begin{cases}0 & -L\le x\le +L\\ \infty & \text{elsewhere}\end{cases}$, is given by: **[BHU 2016]**

(a) $\sqrt{\dfrac{2}{L}}\sin\dfrac{\pi x}{L}$ (b) $\sqrt{\dfrac{2}{L}}\cos\dfrac{\pi x}{L}$ (c) $\sqrt{\dfrac{1}{L}}\sin\dfrac{\pi x}{2L}$ (d) $\sqrt{\dfrac{1}{L}}\cos\dfrac{\pi x}{2L}$

**Soln.** The ground state of a particle in a box is

$$\psi(x)=\sqrt{\frac{2}{2L}}\sin\left(\frac{\pi x}{2L}\right)$$

where $V(x)=\begin{cases}0 \ ; & 0\le x\le 2L\\ \infty \ ; & \text{otherwise}\end{cases}$

Shifting $x\to x-L$, we have

$$V(x)=\begin{cases}0 \ ; & -L\le x\le L\\ \infty \ ; & \text{otherwise}\end{cases} \text{ and}$$

$$\psi(x)=\sqrt{\frac{2}{2L}}\sin\left[\frac{\pi}{2L}(x-L)\right]$$

$$=\sqrt{\frac{1}{L}}\sin\left(\frac{\pi x}{2L}-\frac{\pi}{2}\right)$$

$$=\sqrt{\frac{1}{L}}\cos\left(\frac{\pi x}{2L}\right)$$

**Correct option is (d)**

**Example 31.** A particle is confined in a one-dimensional box of length $l$. Its ground state wave function is given by

$$\psi(x)=\begin{cases}\sqrt{\dfrac{2}{l}}\sin\left(\dfrac{\pi x}{l}\right) & ; \text{ for } 0<x<l\\ 0 & ; \text{ for } \quad x>l\end{cases}$$

The expectation value of the momentum operator with respect to the above wave function is equal to

(a) 1 (b) 0.5 (c) $l/2$ (d) 0 **[HCU 2011]**

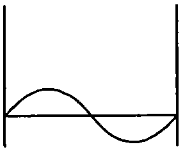
**Soln.** Given: $\psi(x)=\begin{cases}\sqrt{\dfrac{2}{l}}\sin\left(\dfrac{\pi x}{l}\right) & ; \ 0<x<l\\ 0 & ; \ 0>x>l\end{cases}$

$$\therefore \ \langle\hat{p}_x\rangle=\int_0^l\psi^*(x)\left(-i\hbar\frac{d}{dx}\right)\psi(x)\,dx=(-i\hbar)\frac{2\pi}{l^2}\int_0^l\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi x}{l}\right)dx=0$$

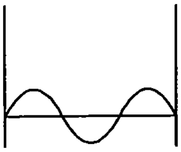
**Correct option is (d)**

**Example 32.** Which of the following represents the first excited state wave function of a massive particle confined to move in a one-dimensional box of length L (the x-axis represents the space co-ordinate and the y-axixrepresents the shape of the wave-function, in each figure.). **[HCU 2012]**

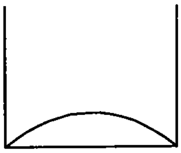
(a)



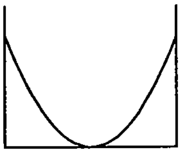
(b)



(c)



(d)

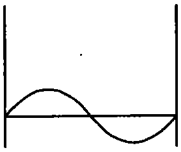


**Soln.** The wave function of the particle inside a box is given by

$$\psi = \sqrt{\frac{2}{L}}\sin\frac{n\pi}{L}x$$

So the wave function of the first excited state

$$\psi = \sqrt{\frac{2}{L}}\sin\frac{2\pi}{L}x$$



**Correct option is (a)**

**Another Method :**

The number of nodes of the *n*th state is $(n-1)$.

So the number of nodes in the 2nd state will be $(2-1)=1$

**Correct option is (a)** .

**Example 33.** If a particle of mass $1.67\times10^{-27}$ kg is confined to move in a box of length $10^{-14}$ m and infinite depth, the minimum kinetic energy of the particle would be **[HCU 2012]**

(a) 0.21 MeV (b) 0.33 MeV (c) 2.03 eV (d) 3.3 eV

**Soln.** The minimum kinetic energy would be

$$E_k = \frac{\pi^2\hbar^2}{2ma^2} = \frac{\pi^2(1.05\times10^{-34})^2}{2\times1.67\times10^{-27}\times(10^{-14})^2} = 3.25\times10^{-13}\text{ J} = 2.03\times10^6\text{ eV} = 2.03\text{ MeV}$$

**Correct option is (c)**

**Example 34.** In a one dimensional potential well with impenetrable walls occupying the domain $0\le x\le d$, a particle in the lowest energy state has the maximum probability to be found at **[HCU 2013]**

(a) $x=0$ (b) $x=d/2$ (c) $x=d/4$ (d) $x=d$

**Soln.** The ground state corresponding to the given potential is $\psi = \sqrt{\frac{2}{d}}\sin\frac{\pi}{d}x$

The probability density $p(x)=\psi^*\psi=\frac{2}{d}\sin^2\frac{\pi}{d}x$.

$p(x)$ will be maximum, when $\frac{dp}{dx}=0$

$$\Rightarrow \frac{2}{d}\frac{\pi}{d}2\cdot\sin\frac{\pi}{d}x\cos\frac{\pi}{d}x=0 \Rightarrow \sin\frac{2\pi}{d}x=\pi$$

$$\Rightarrow \frac{2\pi}{d}x=\pi \qquad \Rightarrow x=\frac{d}{2}$$

**Correct option is (b)**

**Example 35.** A rigid-walled box that extends from –L to +L is divided into three sections by interior walls at –x to +x where x < L. Each section contains one particle of same mass in its ground state. The total energy of the system in the ground state is **[HCU 2014]**

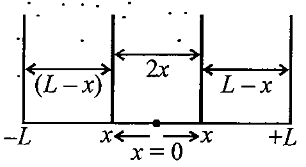
(a) $E_T = \frac{\pi^2\hbar^2}{2m}\left[\frac{2}{(L-x)^2}+\frac{1}{4x^2}\right]$ (b) $E_T = \frac{\pi^2\hbar^2}{6mL^2}$

(c) $E_T = \frac{\pi^2\hbar^2}{2mL^2}\left[\frac{2}{(L+x)^2}+\frac{1}{4x^2}\right]$ (d) $E_T = \frac{\pi^2\hbar^2}{2m}\left[\frac{2}{(L+x)^2}+\frac{1}{(L-x)^2}\right]$

**Soln.** The total energy of system in the ground state is

$$E = \frac{\pi^2\hbar^2}{2m}\left[\frac{1}{(L-x)^2}+\frac{1}{(2x)^2}+\frac{1}{(L-x)^2}\right]$$

$$= \frac{\pi^2\hbar^2}{2m}\left[\frac{2}{(L-x)^2}+\frac{1}{4x^2}\right]$$



**Correct option is (a)**

**Example 36.** For a wave function defined as $\psi(x) = \left(\frac{2}{L}\right)^{1/2}\sin\left(\frac{\pi x}{L}\right)$ in the region $0 < x < L$ and $\psi(x) = 0$ outside this region, the average value of the square of the momentum $(p^2)$ is, **[DU 2016]**

(a) $\frac{\hbar\pi}{L}$ (b) $\frac{h^2\pi^2}{L^2}$ (c) $\frac{\hbar^2\pi^2}{L^2}$ (d) 0

**Soln.** Given : $\psi(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)$

$$\therefore \ \langle p^2\rangle = \int_0^L \psi^*(x)\left(-i\hbar\frac{d}{dx}\right)^2\psi(x)\,dx$$

$$= -\frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)\left(-i\hbar\frac{\pi}{L}\right)^2\sin\left(\frac{\pi x}{L}\right)dx$$

$$= \frac{2\hbar^2\pi^2}{2L^3}\int_0^L\left(1-\cos\frac{2\pi x}{L}\right)dx$$

$$= \frac{\hbar^2\pi^2}{L^2}$$

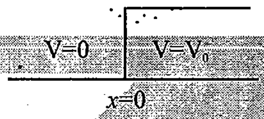
**Correct option is (c)**

# Potential Step and Potential Barrier

## 6.1. Single Step Potential:

The potential function $V(x)$ in a one-dimensional single step potential undergoes a discontinuous change as shown in figure. The potential function $V(x)$ of the potential step may be expressed as,

$$V(x)=\begin{cases}0, & \text{for } x<0\\ V_0, & \text{for } x>0\end{cases}$$



The potential difference between the dees of a cyclotron or, the potential barrier at the surface of a metal is approximately similar to this step potential.

Let a particle of mass $m$ and kinetic energy $E$ is travelling parallel to the $x$-axis and is incident on the potential step at $x = 0$. Since $V(x) = 0$ for $x < 0$ and $V(x) = V_0$ for $x > 0$, the energy E of the particle is wholly kinetic for $x < 0$ and partly kinetic and partly potential for $x > 0$.

From classical point of view in region I, the particle can move freely as $V(x) = 0$. But in region II, if $E < V_0$, the particle will remain in region I for ever and sometimes will be reflected back at $x = 0$. But for $E > V_0$ the particle will not be reflected back at $x = 0$ instead enter the region II.

Quantum mechanically, it will be shown that the electrons will be partly reflected and partly transmitted at the discontinuity.

**Case A: ($E > V_0$)**

The time-independent Schrodinger equation for region I is,

$$-\frac{h^2}{2m}\frac{d^2\psi_1}{dx^2}=E\psi_1 \quad\Rightarrow\quad \frac{d^2\psi_1}{dx^2}+\frac{2mE}{h^2}\psi_1=0 \;\Rightarrow\; \frac{d^2\psi_1}{dx^2}+\frac{2mE}{h^2}\psi_1=0 \qquad \text{... (1)}$$

where, $\alpha=\sqrt{\frac{2mE}{\hbar^2}}$ and $\psi_1$ is the wave function for the particle moving in region I.

For region II, the time-independent Schrodinger equation is,

$$\frac{d^2\psi_2}{dx^2}+\frac{2m}{\hbar^2}(E-V_0)\psi_2=0 \quad\Rightarrow\quad \frac{d^2\psi_2}{dx^2}+\beta^2\psi_2=0 \qquad \text{...(2)}$$

where, $\beta=\sqrt{\frac{2m(E-V_0)}{\hbar^2}}$ and $\psi_2$ is the wave function for the particle in region II.

The general solutions of equations are

$$\psi_1=Ae^{i\alpha x}+Be^{-i\alpha x}, \quad \psi_2=Ce^{i\beta x}+De^{-i\beta x} \qquad \text{...(3)}$$

The terms $e^{i\alpha x}$ and $e^{-i\alpha x}$ represent respectively a wave travelling along the positive $x$-direction and a wave travelling along the negative $x$-direction, i.e., a reflected wave. Again, the terms $e^{i\beta x}$ and $e^{-i\beta x}$ represent respectively a wave travelling along the positive $x$-direction and a wave travelling along the negative $x$-direction (a reflected wave). But due to the presence of only one discontinuity at $x = 0$, there cannot be any reflected waves in region II. Hence the term $De^{-i\beta x}$ is discarded.

$$\therefore \qquad \psi_2 = Ce^{i\beta x} \qquad \text{...(4)}$$

The values of the constants $A$, $B$ and $C$ can be determined by applying the following boundary conditions:

$$\textbf{1. } (\psi_1)_{x=0} = (\psi_2)_{x=0} \text{ and } \quad \textbf{2. } \left(\frac{d\psi_1}{dx}\right)_{x=0} = \left(\frac{d\psi_2}{dx}\right)_{x=0} \qquad \text{....(5)}$$

Using the above boundary condition in equations we get,

$$A + B = C$$

and $\alpha A - \alpha B = \beta C$ or, $\alpha A - \alpha B = \beta(A + B)$

$$\Rightarrow \quad (\alpha - \beta)A = (\alpha + \beta)B \quad \Rightarrow \quad B = \frac{\alpha - \beta}{\alpha + \beta}A \qquad \text{...(6)}$$

Now from equations we get

$$A + \frac{\alpha - \beta}{\alpha + \beta}A = C \quad \Rightarrow \quad C = \left(\frac{2\alpha}{\alpha + \beta}\right)A$$

$$\therefore \qquad \psi_1 = A\left[e^{i\alpha x} + \left(\frac{\alpha - \beta}{\alpha + \beta}\right)e^{-i\alpha x}\right] \qquad \text{...(7)}$$

$$\text{and} \qquad \psi_2 = \left(\frac{2\alpha}{\alpha + \beta}\right)Ae^{i\alpha_2 x} \qquad \text{...(8)}$$

The probability current density for the incident beam of particles in region I is,

$$J_i = \frac{\hbar\alpha}{m}(Ae^{i\alpha x})(Ae^{i\alpha x})^* = \frac{\hbar\alpha}{m}(AA^* e^{i\alpha x}\, e^{-i\alpha x}) \quad \text{or,} \quad J_i = \frac{\hbar\alpha}{m}|A|^2 \qquad \text{...(9)}$$

The probability current density for the reflected beam is,

$$J_r = \frac{\hbar\alpha}{m}(Be^{-i\alpha x})(Be^{-i\alpha x})^* = \frac{\hbar\alpha}{m}|B|^2 \qquad \text{...(10)}$$

The net probability current density in the direction from left to right is

$$J = J_i - J_r = \frac{\hbar\alpha}{m}(|A|^2 - |B|^2) \qquad \text{...(11)}$$

But, $|B|^2 = \dfrac{(\alpha - \beta)^2}{(\alpha + \beta)^2}|A|^2$ [from equation (6)]

$$\therefore \qquad J = \frac{\hbar\alpha}{m}|A|^2\left[1 - \frac{(\alpha - \beta)^2}{(\alpha + \beta)^2}\right] = \frac{\hbar\alpha}{m}|A|^2\left[\frac{4\alpha\beta}{(\alpha + \beta)^2}\right] \quad \Rightarrow \quad J = \frac{4\alpha\beta}{(\alpha + \beta)^2}\frac{\hbar\alpha}{m}|A|^2 \qquad \text{....(12)}$$

Again, the probability current density for the transmitted beam of particles in region II is,

$$J_t = \frac{\hbar\beta}{m}\psi_2\psi_2^* = \frac{\hbar\beta}{m}(Ce^{i\beta x})(Ce^{i\beta x})^* = \frac{\hbar\beta}{m}|C|^2 \qquad ...(13)$$

But, $C = \dfrac{2\alpha}{\alpha+\beta}A$ ...(14)

$$\therefore \quad J_t = \frac{\hbar\beta}{m}\left(\frac{2\alpha}{\alpha+\beta}\right)^2 |A|^2 = \frac{4\alpha^2}{(\alpha+\beta)^2}\frac{\hbar\beta}{m}|A|^2 \qquad ...(15)$$

From equations $J = J_t$, i.e. the net probability density from left to right in region I is equal to the probability density in region II.

$$\text{Again,} \quad J_r + J_t = \frac{\hbar\alpha}{m}|A|^2\frac{(\alpha-\beta)^2}{(\alpha+\beta)^2} + \frac{4\alpha}{(\alpha+\beta)^2}\frac{\hbar\beta}{m}|A|^2 = \left[\frac{(\alpha-\beta)^2}{(\alpha+\beta)^2} + \frac{4\alpha\beta}{(\alpha+\beta)^2}\right]\frac{\hbar\alpha}{m}|A|^2$$

$$= \left[\frac{(\alpha-\beta)^2}{(\alpha+\beta)^2} + \frac{4\alpha\beta}{(\alpha+\beta)^2}\right]J_i \Rightarrow J_r + J_t = J_i \qquad ...(16)$$

This shows that the total number of particles is conserved in the process.

**Reflection and Transmission Coefficients:**

The reflection coefficient is defined as,

$$R = \frac{\text{Reflected flux}\,(J_r)}{\text{Incident flux}\,(J_i)} = \frac{\frac{\hbar\alpha}{m}|B|^2}{\frac{\hbar\alpha}{m}|A|^2} = \frac{|B|^2}{|A|^2} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 \left[\because B = \frac{\alpha-\beta}{\alpha+\beta}A\right] \qquad ...(17)$$

The transmission coefficient is,

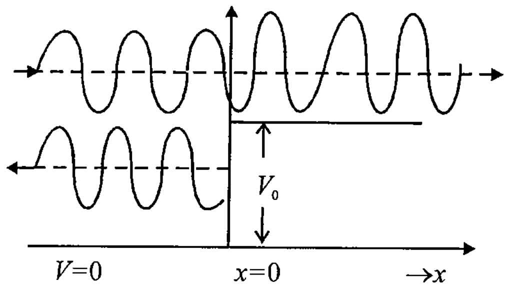
$$T = \frac{\text{Transmitted flux}\,(J_t)}{\text{Incident flux}\,(J_i)} = \frac{\frac{\hbar\beta}{m}|C|^2}{\frac{\hbar\alpha}{m}|A|^2} = \frac{\beta|C|^2}{\alpha|A|^2} \qquad ...(18)$$

$$\frac{\beta}{\alpha}\frac{4\alpha^2}{(\alpha+\beta)^2} = \frac{4\alpha\beta}{(\alpha+\beta)^2}\left[\because C = \frac{2\alpha}{\alpha+\beta}A\right]$$

$$\therefore \quad T + R = \frac{4\alpha\beta}{(\alpha+\beta)^2} + \frac{\alpha-\beta}{(\alpha+\beta)^2} = 1 \qquad ...(19)$$

Equation indicates that at the boundary between regions I and II, the probability of finding the particle is conserved.

Equation shows, that, $|C| > |A|$ as $\alpha > \beta$. In other words, the amplitude ($C$) of the transmitted wave is greater than that of the incident wave (i.e., $A$). The de Broglie wavelength in region I will be shorter than that in region II. This is due to the fact that the kinetic energy of the particle in region I is greater than the kinetic energy is region II. The nature of the wave forms is shown in figure.



The analysis thus shows that, for $E > V_0$, the incident particle has certain probability of being reflected and certain probability of being transmitted.

**Case B : ($E < V_0$)**

For region I ($x<0$), the Schrodinger wave equation is,

$$\frac{d^2\psi_1}{dx^2}+\alpha^2 x=0\text{, where } \alpha^2=\frac{2mE}{\hbar^2} \quad ...(20)$$

The general solution of the above equation is as above,

$$\psi_1 = Ae^{i\alpha x}+Be^{-i\alpha x} \quad ...(21)$$

For region II ($x > 0$), the wave equation is,

$$\frac{d^2\psi_2}{dx^2}-\frac{2m}{\hbar^2}(V_0-E)\psi_2=0\ (\because V_0>E) \quad \text{or,} \quad \frac{d^2\psi_2}{dx^2}-\gamma^2 x=0\left[\gamma^2=\frac{2m}{\hbar^2}(V_0-E)\right] \quad ...(22)$$

The general solution of the above equation is, $\psi_2 = Ce^{\gamma x}+De^{-\gamma x}$ ...(23)

But the term, $e^{\gamma x}\rightarrow\infty$ as $x\rightarrow 0$ which contradicts the condition of well-behaved wave function.

$$\therefore \quad \psi_2 = De^{-\gamma x} \quad ...(24)$$

The values of the constants A, B and D can be calculated by applying the following boundary conditions:

$$\textbf{1. } (\psi_1)_{x=0}=(\psi_2)_{x=0} \text{ and } \quad \textbf{2. } \left(\frac{d\psi_1}{dx}\right)_{x=0}=\left(\frac{d\psi_2}{dx}\right)_{x=0} \quad ...(25)$$

Thus we get, $A + B = D$ and $i\alpha A - i\alpha B = -\gamma D$

Solving we get, $i\alpha(A-B)=-\gamma(A+B) \Rightarrow (i\alpha+\gamma)A=(i\alpha-\gamma)B$

$$\Rightarrow \quad B=\frac{i\alpha+\gamma}{i\alpha-\gamma}A \quad ...(26)$$

$$\text{and} \quad D = A+\frac{i\alpha+\gamma}{i\alpha-\gamma}A=\left(\frac{i\alpha-\gamma+i\alpha+\gamma}{i\alpha-\gamma}\right)A \Rightarrow D=\frac{2i\alpha}{i\alpha-\gamma}A \quad ...(27)$$

Now from equation (26), $BB^*=\left(\frac{i\alpha+\gamma}{i\alpha-\gamma}\right)A\left(\frac{i\alpha+\gamma}{i\alpha-\gamma}\right)A^* = \left(\frac{i\alpha+\gamma}{i\alpha-\gamma}\right)\left(\frac{-i\alpha+\gamma}{-i\alpha-\gamma}\right)AA^*$

$$\Rightarrow \quad BB^*=AA^* \Rightarrow |B|^2=|A|^2 \quad ...(27)$$

Thus the probability current in region I is zero. Equation shows that the intensity of the incident wave is equal to the intensity of the reflected wave. Hence the incident and reflected probability currents cancel each other. This can be proved as follows:

We have the probability current density in region II as,

$$J_t = -\frac{i\hbar}{2m}\left[\psi_2^*\frac{d\psi_2}{dx} - \psi_2\frac{d\psi_2^*}{dx}\right] = -\frac{i\hbar}{2m}[D^*e^{-\gamma x}D(-\gamma)e^{-\gamma x} - De^{-\gamma x}.D^x(-\gamma)e^{-\gamma x}]$$

$$\Rightarrow \quad J_t = \frac{i\hbar}{2m}DD^*\gamma[e^{-2\gamma x} - e^{-2\gamma x}] = 0 \qquad ...(28)$$

**Reflection and Transmission Coefficients:**

The reflection coefficient, $R = \frac{J_r}{J_i} = \frac{(Be^{-i\alpha x})(Be^{-i\alpha x})^*\frac{\hbar\alpha}{m}}{(Ae^{i\alpha x})(Ae^{i\alpha x})^*\frac{\hbar\alpha}{m}} = \frac{BB^*}{AA^*} = \frac{|B|^2}{|A|^2}$ ... (29)

Since $|B|^2 = |A|^2$ for $V_0 > E$,

$\therefore \quad R = 1$

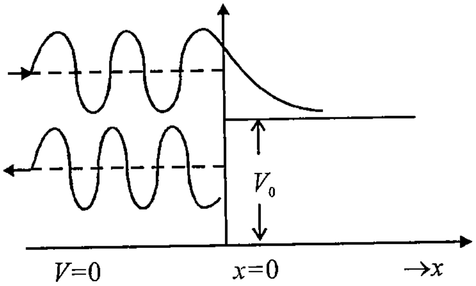
Hence, the incident wave is totally reflected at the potential step at $x = 0$.

The transmission coefficient, $T = \frac{J_t}{J_i}$, but for region II, $J_t = 0$

$\therefore \quad T = 0$

The above analysis leads to the following conclusions:

1. The incident wave is totally reflected at the potential step.
2. Both in region I and II, the probability current is zero.
3. In region II, the wave function $\psi_2 = De^{\gamma x}$, i.e. $\psi_2$ is exponentially damped. This leads to a finite probability of finding the particle in region II.



The phenomenon is very much similar to the case of total internal reflection of electromagnetic waves while travelling from denser to a rarer medium.

If for finite energy $E$, the potential $V_0 \to \infty$, then $\gamma = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} \to \infty$

The wave function $\psi_2$ in this case is, $\psi_2 = De^{-\gamma x} \to De^{-\alpha x} \to 0$

Again, $B=\left(\dfrac{i\alpha+\gamma}{i\alpha-\gamma}\right)A$ and $D=\dfrac{2i\alpha}{i\alpha-\gamma}A$

$$\therefore \quad \lim_{\gamma\to\infty}\frac{B}{A}=\lim_{\gamma\to 0}\frac{\alpha-i\gamma}{\alpha+i\gamma}=\lim_{\gamma\to 0}\frac{\dfrac{\alpha}{\gamma}-i}{\dfrac{\alpha}{\gamma}+i}=\frac{-i}{i}=-1\Rightarrow\frac{B}{A}=-1 \text{ or, } A+B=0$$

Also, $\displaystyle\lim_{\gamma\to\infty}\frac{D}{A}=\frac{2i\alpha}{i\alpha-\gamma}=\frac{2\alpha}{\alpha+i\gamma}=0$

$\therefore \quad D=0$

The wave function in region I is, $\psi_1(x)=Ae^{i\alpha x}+Be^{-i\alpha x}$

When $x=0$, $\psi_1(0)=A+B=0$ and the wave function in region II is, $\psi_2(x)=De^{-\gamma x}$ at $x=0$, $\psi_2(0)=D=0$

The slope of $\psi_1$ at $x=0$ is, $\left(\dfrac{d\psi_1}{dx}\right)_{x=0}=A(i\alpha)e^{i\alpha 0}+B(-i\alpha)e^{-i\alpha 0}$

$$=i\alpha(A-B)=i\alpha[A-(-A)]=2i\alpha A[\because A+B=0]$$

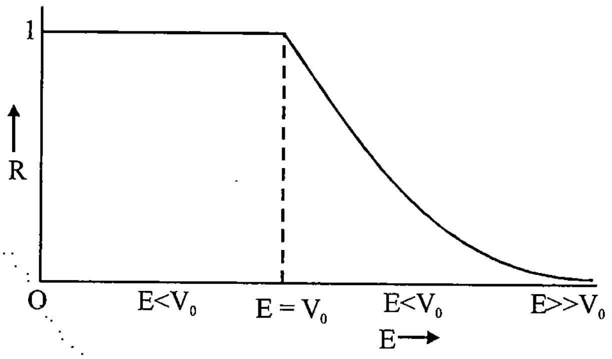
Hence, at $x=0$, where the value of $V(x)$ suddenly increases from 0 to $\infty$, the wave function $\psi_1(0)$ becomes zero. Also, the slope of $\psi_1$ at $x=0$, i.e., $\left(\dfrac{d\psi}{dx}\right)_{x=0}$ suddenly decreases from $2i\alpha A$ to zero.

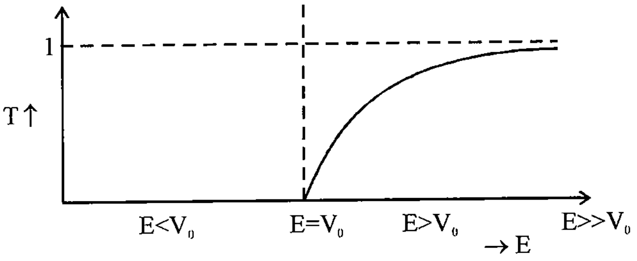
**Penetration depth:**

It is defined as the distance at which wave function reduces to 1/e of the value of the wave function at the barrier (x = 0) and given by

$$x=\frac{1}{\beta}=\sqrt{\frac{\hbar^2}{2m(V_0-E)}}$$

The fact that particle can penetrate into the classically forbidden region leads to Quantum Mechanical Tunnelling. If the potential barrier is infinitely high i.e. $V_0\to\infty$, then penetration depth becomes very much negligible.





**Example 1.** Electrons of energy 3.8 eV are subjected to a potential $V = 0$ for $x < 0$ and $V = 7.6$ eV for $x > 0$. (a) Find the wavelength of the sinusoidal variation of the wave function in the classically allowed region (b) the reflected wave differs from the incident wave by a phase factor. Find this phase factor.

**Soln.** **(a)** The wave number 'k' is $\alpha = \sqrt{\dfrac{2m_e E}{\hbar^2}} = \sqrt{\dfrac{3.8\,eV}{38\times 10^{-3} eV\ nm^2}} = 10\ nm^{-1}$

The wavelength is $\lambda = \dfrac{2\pi}{\alpha} = \dfrac{6.28}{10\ nm^{-1}} \approx 0.63\ nm$

(b) The wave function in the classically allowed region is

$$\psi = A_1\left[e^{i\alpha x} + \frac{i\alpha+\beta}{i\alpha-\beta}e^{-i\alpha x}\right]$$

Thus the reflected wave differs from the incident wave by a phase factor $f = \dfrac{ik+\gamma}{ik-\gamma}$

We have $\alpha = 10\ nm^{-1}$ and similarly $\beta = \sqrt{\dfrac{2m(V_0-E)}{\hbar^2}} = 10\ nm^{-1}$. So, $f = e^{i\frac{3\pi}{2}}$

**Example 2.** Electrons of energy 10 eV are incident on a potential step of height 13.8 eV. Find the distance in which the probability density of finding the particle decreases to a factor of 0.01 as it penetrates into the classically forbidden region.

**Soln.** The wave function in the classically forbidden region is of the form

$$\psi = Ce^{-\beta x}$$

where, $\beta = \sqrt{\dfrac{2m(V_0-E)}{\hbar^2}} = \sqrt{\dfrac{(13.8-10)eV}{38\times 10^{-3}\ eV\ nm^2}} = 10\ nm^{-1}$

According to the question,

$$P(x+d) = 0.01P(x) \Rightarrow |C|^2 e^{-2\gamma(x+d)} = 0.01|C|^2 e^{-2\gamma x}$$

$$\Rightarrow e^{2\gamma d} = 100 \Rightarrow 2\gamma d = \ln(100) \Rightarrow d = \frac{\ln 10}{\gamma} = 0.23\ nm$$

**Example 3.** An electron beam is incident on a potential barrier of height 0.03 eV and of infinite width. Find the reflection coefficient and transmission coefficient at the barrier if the energy of the incident electron beam be (a) 0.04 eV, (b) 0.025 eV.

**Soln.** **(a)** In this case, $E > V_0$.

Reflection coefficient $R = \left(\dfrac{\sqrt{E} - \sqrt{(E-V_0}}{\sqrt{E} + \sqrt{E-V_0}}\right)^2 \approx 0.1$

Transmission coefficient $T = 1 - R = 1 - 0.1 = 0.9$

**(b)** In this case, $E < V_0$. So, $T = 0$ and $R = 1$, i.e. all the electrons that are incident on the barrier are reflected back and no electron will be transmitted through the barrier.

## 6.2. One-Dimensional Rectangular Potential Barrier (Quantum Mechanical Tunnelling Effect):

The diagrammatic representation of a rectangular potential barrier is shown in figure which extends from $x = 0$ to $x = a$. The potential of the barrier in this region is constant and equal to $V_0$. On the left and right side of the barrier the potential $V = 0$.

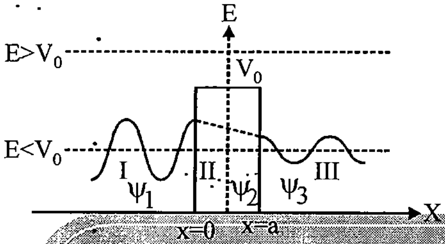


**Fig.** One-dimensional rectangular potential barrier

Now, we suppose that a stream of particles of energy E be incident from left on the barrier surface at $x = 0$.

The following two cases arise:

1. If $E > V_0$, then according to classical mechanics the particles will be wholly transmitted and no reflection is possible. But quantum mechanically there is always some probability at $x = 0$ and $x = a$.
2. If $E < V_0$, then classically the particles will be wholly reflected and hence penetration through the barrier is impossible. But quantum mechanically there is always some probability of penetration into the barrier and appearance of the particles in region III. This finite probability of transmission through the barrier even $E<V_0$ is called the quantum mechanical tunnelling effect.

**Case A: When $E > V_0$**

For region I, ($x<0$) where $V(x) = 0$, the time-independent Schrodinger equation is,

$$\frac{d^2\psi_1}{dx^2}+\frac{2m}{\hbar^2}E\psi_1=0 \quad\Rightarrow\quad \frac{d^2\psi_1}{dx^2}+\alpha^2\psi_1=0 \qquad ....(1)$$

where, $\alpha^2=\dfrac{2mE}{\hbar^2}$ and $\psi_1$ is the wave function in region I.

The general solution of equation is, $\psi_1(x)=Ae^{i\alpha x}+Be^{-i\alpha x}$ ....(2)

For region II, ($0 < x < a$), where $V(x)=V_0$, the Schrodinger equation is,

$$\frac{d^2\psi_2}{dx^2}+\frac{2m}{\hbar^2}(E-V_0)\psi_2=0 \quad\Rightarrow\quad \frac{d^2\psi_2}{dx^2}+\beta^2\psi_2=0 \qquad ....(3)$$

where, $\beta^2=\dfrac{2m(E-V_0)}{\hbar^2}$ and $\psi_2$ is the wave function in region II.

The general solution of equation is, $\psi_2(x)=Ce^{i\beta x}+De^{-i\beta x}$ ....(4)

Here, $De^{-i\beta x}$ represents the wave which is reflected at the boundary at $x = a$.

For region III, $(x > a)$, where $V(x) = 0$ again, the Schrodinger equation is,

$$\frac{d^2\psi_3}{dx^2} + \alpha^2\psi_3 = 0 \text{ and the general solution is, } \psi_3(x) = Fe^{i\alpha x} \qquad \text{.... (5)}$$

Since in region III, the wave propagates to the right only, there is no question of reflection and as such the $e^{-i\alpha x}$ term is absent from equation.

The five constants $A$, $B$, $C$, $D$ and $F$ can be obtained by applying the boundary conditions at $x = 0$ and $x = a$.

The boundary conditions are:

1. $\psi$ is continuous at $x = 0$ and $x = a$ as the probability density must be continuous, and
2. $\frac{d\psi}{dx}$ at $x = 0$ and $x = a$ is continuous since probability current must be continuous.

Applying the boundary conditions at $x = 0$ and $x = a$ we get,

$$A + B = C + D,\ A - B = \frac{B}{\alpha}(C - D),\ Ce^{i\beta a} + De^{-i\beta a} = Fe^{i\alpha a},\ Ce^{i\beta a} - De^{-i\beta a} = \frac{\alpha}{\beta}e^{i\alpha a}$$

Now solving equations we get, $A = \frac{C}{2}\left(1+\frac{\beta}{\alpha}\right) + \frac{D}{2}\left(1-\frac{\beta}{\alpha}\right)$, $B = \frac{C}{2}\left(1-\frac{\beta}{\alpha}\right) + \frac{D}{2}\left(1+\frac{\beta}{\alpha}\right)$ ....(5)

Solving equations we get, $C = \frac{F}{2}\left(1+\frac{\alpha}{\beta}\right)e^{i(\alpha-\beta)a}$, $D = \frac{F}{2}\left(1-\frac{\alpha}{\beta}\right)e^{i(\alpha+\beta)a}$ ....(6)

Substituting the values of C and D from equation (6) in equation (5) we get,

$$A = \frac{F}{4}e^{i\alpha a}\left[\left(1+\frac{\beta}{\alpha}\right)\left(1+\frac{\alpha}{\beta}\right)e^{-i\beta a} + \left(1-\frac{\beta}{\alpha}\right)\left(1-\frac{\alpha}{\beta}\right)e^{i\beta a}\right] \qquad \text{....(7)}$$

$$B = \frac{F}{4}e^{i\alpha a}\left[\left(1-\frac{\beta}{\alpha}\right)\left(1+\frac{\alpha}{\beta}\right)e^{-i\beta a} + \left(1+\frac{\beta}{\alpha}\right)\left(1-\frac{\alpha}{\beta}\right)e^{i\beta a}\right] \qquad \text{....(8)}$$

From equation
$$\frac{F}{A} = \frac{4e^{-i\alpha a}}{\left(1+\frac{\beta}{\alpha}\right)\left(1+\frac{\alpha}{\beta}\right)e^{-i\beta a} + \left(1-\frac{\beta}{\alpha}\right)\left(1-\frac{\alpha}{\beta}\right)e^{i\beta a}}$$

$$= \frac{2\alpha\beta e^{-i\alpha a}}{(\alpha^2+\beta^2)\left(\frac{e^{-i\beta a} - e^{i\beta a}}{2}\right) + 2\alpha\beta\left(\frac{e^{-i\beta a} + e^{i\beta a}}{2}\right)}$$

$$= \frac{2\alpha\beta e^{-i\alpha a}}{2\alpha\beta\cos\beta a - i(\alpha^2+\beta^2)\sin\beta a}$$

Transmission co-efficient $T = \frac{J_t}{J_i} = \frac{\frac{\hbar k}{m}(Fe^{iax})(Fe^{-iax})^*}{\frac{\hbar k}{m}(Ae^{i\alpha x})(Ae^{i\alpha x})^*} = \frac{FF^*}{AA^*} = \frac{|F|^2}{|A|^2} = \left(\frac{F}{A}\right)\left(\frac{F}{A}\right)^*$

$$= \frac{4\alpha^2\beta^2}{4\alpha^2\beta^2\cos^2\beta a + (\alpha^2+\beta^2)^2\sin^2\beta a} = \frac{1}{1+\frac{(\alpha^2-\beta^2)^2}{4\alpha^2\beta^2}\sin^2\beta a} \qquad ....(9)$$

But, $\alpha^2 - \beta^2 = \frac{2mE}{\hbar^2} - \frac{2m}{\hbar^2}(E - V_0) \Rightarrow (\alpha^2-\beta^2)^2 = \left(\frac{2m}{\hbar^2}\right)^2 V_0^2$

and $\alpha^2\beta^2 = \left(\frac{2mE}{\hbar^2}\right)\left[\frac{2m}{\hbar^2}(E-V_0)\right] = \left(\frac{2m}{\hbar^2}\right)^2 E(E-V_0)$

$$\therefore \quad T = \frac{1}{1+\frac{V_0^2}{4E(E-V_0)}\sin^2\beta a} \qquad ....(10)$$

From equation, we note that T = 1, if $\sin^2\beta a = 0$ or, $\beta a = n\pi$, $n = 1, 2, 3, ......$, then,

$$a = \frac{n\pi}{\beta} = n\frac{\pi\hbar}{\beta\hbar} = n\frac{\frac{h}{2}}{p} = n\frac{\lambda_2}{2}$$

where, $\hbar = \frac{h}{2\pi}$ and $B\hbar = p$, the momentum of the particle in region II, $\lambda_2 = \frac{h}{p}$, the de-Broglie wavelength of the particle.

Hence, if the barrier region contains an integral multiple of half de-Broglie wavelengths, then the barrier transmission is complete. This is Ramsauer-Townsend Effect. It explains the phenomenon of scattering of low energy electrons from the atoms of noble gases.

Equation shows that $T$ oscillates for $\frac{E}{V_0} > 1$ and approaches to unity of $E >> V_0$

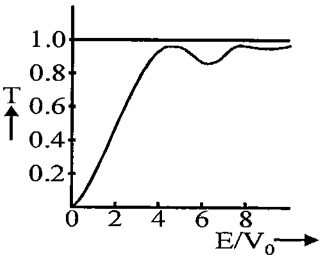


**Fig.** Transmission through a rectangular potential barrier as a function of energy

$$\text{Reflection coefficient } R=\left(\frac{B}{A}\right)\left(\frac{B}{A}\right)^{*}=\frac{\dfrac{(\alpha^2-\beta^2)^2}{4\alpha^2\beta^2}\sin^2\beta a}{1+\left(\dfrac{\alpha^2-\beta^2}{4\alpha^2\beta^2}\right)\sin^2\beta a}=\frac{\dfrac{V_0^2}{4E(E-V_0)}\sin^2\beta a}{1+\dfrac{V_0^2}{4E(E-V_0)}\sin^2\beta a} \quad \text{...(11)}$$

From equations, $R + T = 1$

The natures of the wave functions $\psi_1, \psi_2$ and $\psi_3$ in region I, II and III respectively are shown in figure.

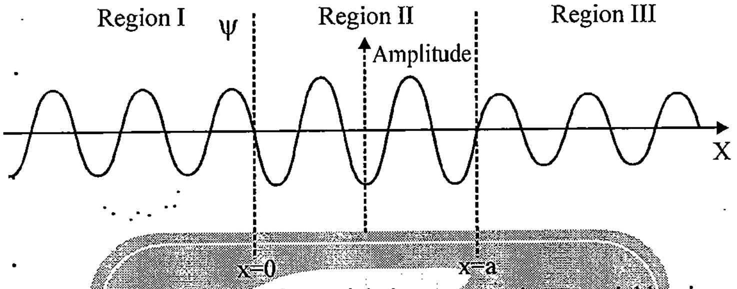


**Fig.** Wave function of a particle in a rectangular potential barrier

**Case B : When $E < V_0$**

The time-independent Schrodinger equations for regions I, II and III are given by,

$$\frac{d^2\psi_1}{dx^2}+\alpha^2\psi_1=0, \quad \frac{d^2\psi_2}{dx^2}-\beta^2\psi_2=0 \quad \text{and} \quad \frac{d^2\psi_3}{dx^2}+\alpha^2\psi_3=0 \quad \text{....(12)}$$

where, $\alpha^2=\dfrac{2mE}{\hbar^2}$ and $\beta^2=\dfrac{2m}{\hbar^2}(V_0-E)$

The solutions of the above three equations for regions I, II and III are respectively,

$$\psi_1=Ae^{i\alpha x}+Be^{-i\alpha x}, \quad \psi_2=Ce^{\beta x}+De^{-\beta x} \quad \text{and} \quad \psi_3=Fe^{i\alpha x}+Ge^{-i\alpha x} \quad \text{....(13)}$$

In region III as the wave propagates along the forward direction, there is no reflection and hence $Ge^{-i\alpha x}$ is discarded.

$$\therefore \quad \psi_3=Fe^{i\alpha x} \quad \text{....(14)}$$

Here $A$, $B$, $C$, $D$ and $F$ are arbitrary constants. To find the values of these constants we apply the following boundary conditions:

**1.** $(\psi_1)_{x=0}=(\psi_2)_{x=0}$ **2.** $\left(\dfrac{d\psi_1}{dx}\right)_{x=0}=\left(\dfrac{d\psi_2}{dx}\right)_{x=0}$ ....(16)

Applying these boundary conditions to equations (13) we get,

$A+B=C+D$ and $i\alpha A-i\alpha B=\beta C-\beta D$

Solving the above two equations we get,

$$A=\left(1-\frac{i\beta}{\alpha}\right)\frac{C}{2}+\left(1+\frac{i\beta}{\alpha}\right)\frac{D}{2} \quad \text{and} \quad B=\left(1+\frac{i\beta}{\alpha}\right)\frac{C}{2}+\left(1-\frac{i\beta}{\alpha}\right)\frac{D}{2} \quad \text{....(17)}$$

Similarly the boundary conditions at $x = a$ are:

$$\textbf{3.}\ (\psi_2)_{x=a} = (\psi_3)_{x=a} \qquad \textbf{4.}\ \left(\frac{d\psi_2}{dx}\right)_{x=a} = \left(\frac{d\psi_3}{dx}\right)_{x=a} \qquad \text{....(18)}$$

Now from equations (13) we get,

$$Ce^{\beta a} + De^{-\beta a} = Fe^{\alpha a} \quad \text{and} \quad \beta Ce^{\beta a} - D\beta e^{-\beta a} = i\alpha Fe^{i\alpha a} \qquad \text{....(19)}$$

Solving equations we get,

$$C = \left(1 + \frac{i\alpha}{\beta}\right)\frac{F}{2}e^{-\beta\alpha}e^{i\alpha a} \ \text{or,}\ C = \left(1 + \frac{i\alpha}{\beta}\right)\frac{F}{2}e^{(i\alpha-\beta)a} \ \text{and}\ D = \left(1 - \frac{i\alpha}{\beta}\right)\frac{F}{2}e^{(i\alpha+\beta)a} \qquad \text{....(20)}$$

Using these values of C and D in equations (17) we get,

$$A = \frac{F}{4}e^{i\alpha a}\left[\left(1 + \frac{i\beta}{\alpha}\right)\left(1 + \frac{\alpha}{i\beta}\right)e^{-\beta a} + \left(1 - \frac{i\beta}{\alpha}\right)\left(1 - \frac{\alpha}{i\beta}\right)e^{\beta a}\right] \qquad \text{....(21)}$$

$$B = \frac{F}{4}e^{i\alpha a}\left[\left(1 - \frac{i\beta}{\alpha}\right)\left(1 + \frac{\alpha}{i\beta}\right)e^{-\beta a} + \left(1 + \frac{i\beta}{\alpha}\right)\left(1 - \frac{\alpha}{i\beta}\right)e^{\beta a}\right] \qquad \text{....(22)}$$

**Transmission Coefficient and Reflection Coefficient:**

The transmission coefficient is given by, $T = \dfrac{J_t}{J_i} = \dfrac{\dfrac{\hbar k}{m}(Fe^{i\alpha x})(Fe^{i\alpha x})^*}{\dfrac{\hbar k}{m}(Ae^{i\alpha x})(Ae^{i\alpha x})} = \left(\dfrac{F}{A}\right)\left(\dfrac{F}{A}\right)^*$ ....(23)

Now from equation we get, $\dfrac{F}{A} = \dfrac{4}{e^{i\alpha a}\left[\left(1 + \dfrac{i\beta}{\alpha}\right)\left(1 + \dfrac{\alpha}{i\beta}\right)e^{-\beta a} + \left(1 - \dfrac{i\beta}{\alpha}\right)\left(1 - \dfrac{\alpha}{i\beta}\right)e^{\beta a}\right]}$

$$\therefore T = \left(\frac{F}{A}\right)\left(\frac{F}{A}\right)^* = \frac{1}{\cosh^2(\beta a) + \frac{1}{4}\left(\frac{\beta}{\alpha} - \frac{\alpha}{\beta}\right)^2 \sinh^2(\beta a)} = \frac{1}{1 + \sinh^2(\beta a) + \left(\frac{\beta^2 - \alpha^2}{2\beta\alpha}\right)^2 \sinh^2(\beta a)}$$

$$= \frac{1}{1 + \sinh^2(\beta a)\left[1 + \left(\frac{\beta^2 - \alpha^2}{2\alpha\beta}\right)^2\right]} = \frac{1}{1 + \left(\frac{\beta^2 + \alpha^2}{4\alpha^2\beta^2}\right)\sinh^2 \beta a} \qquad \text{....(24)}$$

Substituting $\alpha^2 = \dfrac{2mE}{\hbar^2}$ and $\beta^2 = \dfrac{2m(E - V_0)}{\hbar^2}$ we get, from equations (24)

$$T = \frac{1}{1 + \frac{V_0^2}{4E(V_0 - E)}\sinh^2(\beta a)} \qquad \text{....(25)}$$

Reflection coefficient $R = \frac{J_r}{J_i} = \frac{\frac{\hbar k}{m}(Be^{-i\alpha x})\,(Be^{-i\alpha x})^*}{\frac{\hbar k}{m}(Ae^{i\alpha x})\,(Ae^{i\alpha x})^*} = \left(\frac{B}{A}\right)\left(\frac{B}{A}\right)^* = \frac{|B|^2}{|A|^2}$

$$= \frac{\frac{(\beta^2+\alpha^2)^2}{4\alpha^2\beta^2}\sin h^2\beta a}{\cos h^2\beta a + \frac{1}{4}\frac{(\beta^2-\alpha^2)^2}{4\alpha^2\beta^2}} = \frac{\left(\frac{\beta^2+\alpha^2}{2\beta\alpha}\right)^2\sin h^2\beta a}{\cos h^2\beta a + \left(\frac{\beta^2-\alpha^2}{2\beta\alpha}\right)^2\sin h^2\beta a}$$

$$\Rightarrow \quad R = \frac{\left(\frac{\beta^2+\alpha^2}{2\beta\alpha}\right)^2\sin h^2\beta a}{1+\left(\frac{\beta^2+\alpha^2}{2\beta\alpha}\right)^2\sin h^2\beta a} \qquad ....(26)$$

**Quantum Mechanical Tunnelling:**

From equation, we see that the transmission coefficient $T \neq 0$ which means that there is a finite probability of transmission of the particles through the potential barrier of height $V_0$ and width $a$ even if $E < V_0$. This cannot be accounted for in view of classical theory.

Thus quantum mechanical tunnelling is the phenomenon of transmission of a particle through a potential of finite width and height when its energy is less than the barrier height.

The transmission probability depends on $V_0$ and $a$ such that $T$ is small if $V_0$ and $a$ are large.

Again, when $a \rightarrow$ large value $\beta a \rightarrow \infty$.

$$\therefore \quad \sin h^2(\beta a) = \left(\frac{e^{\beta a}-e^{-\beta a}}{2}\right)^2 = \frac{e^{2\beta a}}{4} \gg 1 \qquad .... (27)$$

$$\therefore \quad T = \frac{16\alpha^2\beta^2}{(\beta^2+\alpha^2)}e^{-\beta a} = \frac{16(V_0-E)}{V_0^2}e^{-2\beta a} \qquad ....(28)$$

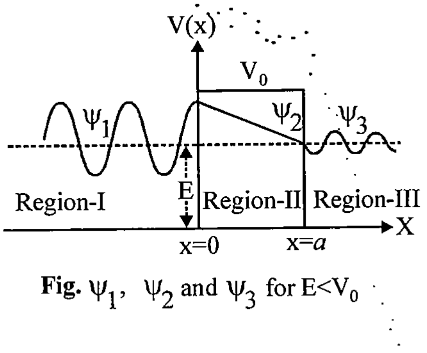


Fig. $\psi_1$, $\psi_2$ and $\psi_3$ for E<V$_0$

The waveform in region I, II and II are shown in figure. The appearance of the incident and transmitted waves are the same as in case $E > V_0$. But for $E < V_0$, there is no wave inside the barrier except an exponentially decaying function as in evident from equation.

From equations it is easily seen that, $R + T = 1$.

From the foregoing discussion the following conclusions are drawn:

**Case A: When $E > V_0$**

1. When $E \to V_0$, $\beta a = a\sqrt{\frac{2m}{\hbar^2}(E-V_0)} \ll 1$

$$\therefore \quad \sin\beta a = a\sqrt{\frac{2m}{\hbar^2}(E-V_0)} \text{ and } \sin^2\beta a = \frac{2m(E-V_0)a^2}{\hbar^2}$$

From equation (25) $T = \dfrac{1}{1+\dfrac{V_0^2}{4E(E-V_0)}\sin^2\beta a} = \dfrac{1}{1+\dfrac{V_0^2}{4E(E-V_0)}\times\dfrac{2m(E-V_0)a^2}{\hbar^2}}$

$$= \frac{1}{1+\dfrac{2mV_0^2a^2}{4E\hbar^2}} \simeq \frac{1}{1+\dfrac{mV_0^2a^2}{2E\hbar^2}},$$

2. If $E > V_0$, then $(E - V_0)$ increases, $\beta = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$ also increases. But $(E - V_0)$ increases more rapidly than $\sin^2\beta a$ for given values of $a$ and $V_0$. Hence $T$ increases with the increase of energy E of the incident particle.

3. In this case as shown before that if $a = n(\lambda/2)$, then there is a perfect transmission of the incident beam of particles.

**Case B : When $E < V_0$**

1. There is a finite probability of transmission of a particle through a potential barrier of width $a$ and height $V_0$ even if $E < V_0$.

2. The factor $(V_0 - E)$ decreases with the increase of E.

$$\therefore \quad \beta = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

also decreases. The rate of decrease of $\sin^2\beta a$ for a fixed value of $a$ and $V_0$ is more than that of $(V_0 - E)$. So $T$ increases steadily with $E$.

3. When $E \to V_0$, $\beta a = a\sqrt{2m(V_0-E)} \ll 1$

$$\therefore \quad \sinh\beta a = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}a \text{ and } \sinh^2\beta a = \frac{2m(V_0-E)a}{\hbar^2}$$

Then, from equation (25), $T = \dfrac{1}{1+\dfrac{V_0^2}{4E(V_0-E)}\times\dfrac{2m(V_0-E)a}{\hbar^2}}$

$$\simeq \frac{1}{1+\frac{2mV_0^2a^2}{4E\hbar}} = \frac{1}{1+\frac{mV_0a^2}{2\hbar^2}} \quad (\text{as } E \to V_0)$$

This value $T$, in case $E > V_0$, is the same as the value of $T$ for $E < V_0$. $T$ is small, if $V_0$ and $a$ are large.

**Example 4:** Calculate the percentage transmission of an electron beam of energy 3eV when incident on a potential barrier of height 4eV and width 20Å.

**Soln.:** Transmission coefficient $T = \frac{16E}{V_0}\left(1-\frac{E}{V_0}\right)e^{-2\beta a}$

Given: $E = 3eV, V_0 = 4\,eV, a = 20A^0$. Here, $2\beta a = 2a \times \sqrt{\frac{2m}{\hbar^2}(V_0 - E)} \approx 20.49$

Therfore, $T = \frac{16E}{V_0}\left(1-\frac{E}{V_0}\right)e^{-2\beta a} = 3.797 \times 10^{-9}$ (approx)

Hence, the % of transmission is = $3.797 \times 10^{-9} \times 100 = 3.797 \times 10^{-7}$.

**Example 5.** An alpha particle is trapped in a nucleus of radius $1.4 \times 10^{-15}$m. What is the probability that it will escape from the nucleus if its energy is 2MeV? The potential barrier at the surface of the nucleus is 4 MeV and the mass of the $\alpha$-particle = $6.64 \times 10^{-27}$ kg

**Soln.** Transmission coefficient $T = \frac{16E}{V_0}\left(1-\frac{E}{V_0}\right)e^{-2\beta a}$

Given: $E = 2MeV, V_0 = 4\,MeV, a = 2 \times 1.4 \times 10^{-15}\,m$

**Example 6.** Transmission probability of a particle with energy $E$ moving through a one-dimensional rectangular potential barrier of height $U_0$ ($U_0 < E$) **[BHU 2011]**

(a) is always 1

(b) is never 1

(c) varies from 0 to 1 periodically with increasing $\frac{E}{U_0}$

(d) becomes 1 periodically with increasing $\frac{E}{U_0}$ but never becomes 0

**Soln.**

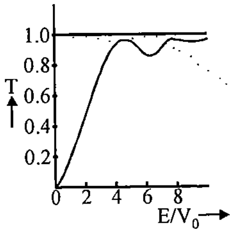


**Fig.** Transmission through a rectangular potential barrier as a function of energy

**Correct option is (d)**

**Example 7.** A free particle with knietic energy $E$ and de Broglie wavelength $\lambda$ enters a region in which it has potential energy $V$. What is the particle's new de Broglie wavelength ? **[BHU 2012]**

(a) $\lambda\left(1-\frac{V}{E}\right)$ (b) $\lambda\left(1-\frac{E}{V}\right)^{-1}$ (c) $\lambda\left(1+\frac{V}{E}\right)^{1/2}$ (d) $\lambda\left(1-\frac{V}{E}\right)^{-1/2}$

**Soln.** The de-Broglie wavelength is $\lambda=\frac{h}{\sqrt{2mE}}$, and the new de-Broglie wavelength $\lambda'$ is

$$\lambda'=\frac{h}{\sqrt{2m(E-V)}}=\frac{h}{\sqrt{2mE\left(1-\frac{V}{E}\right)}}$$

$$=\frac{h}{\sqrt{2mE}}\left(1-\frac{V}{E}\right)^{-1/2}$$

$$=\lambda\left(1-\frac{V}{E}\right)^{-1/2}$$

**Correct option is (d)**

# Chapter 7

# Linear Harmonic Oscillator

If a system vibrates about an equilibrium position such that (a) the magnitude of the force acting on it is proportional to displacement from the equilibrium position and (b) the direction of the force is opposite to that of displacement, it is called a *linear harmonic oscillator* (LHO).

The importance of the LHO in classical as well as quantum physics is that many complex systems can be reduced to a collection of harmonic oscillators of difference frequencies using Fourier analysis. The method is suitable for many-body systems, like the atoms in crystalline solids. In fact in the theory of specific heat of solids, Debye assumed the collective motion of the atoms in the solid as made up of LHO, the frequency of vibration ranging up to a maximum cut-off frequency depending on the nature of the system.

## 7.1. Energy of the LHO According to Classical Mechanics:

Let at any time $t$, the displacement of a particle of mass $m$ constraint to move along the X-axis be $x$ from the equilibrium position $x = 0$. The restoring force on the particle is $F = -kx$ where, $k$ is a positive constant. Then the Newton's equation of motion is,

$$m\frac{d^2x}{dt^2} = -kx \quad \text{or,} \quad \frac{d^2x}{dt^2} + \omega^2 x = 0 \ \left[\text{where, } \omega^2 = \frac{k}{m}\right] \qquad \text{....(1)}$$

The general solution of equation is, $x = a\sin(\omega t + \phi)$ ....(2)

where $a$ and $\psi$ are constants which may be determined from the initial conditions. Equation represents the simple harmonic motion (SHM) of the particle with frequency,

$$\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad \text{....(3)}$$

The potential energy $V(x)$ at displacement $x$ is,

$$V = \int_0^x \frac{dV}{dx}dx = \int_0^x -F dx = -\int_0^x (-kx)dx = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2 \left[\because \omega^2 = \frac{k}{m}\right] \qquad \text{....(4)}$$

The kinetic energy $T$ at $x$ is given by,

$$T = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}ma^2\omega^2\cos^2(\omega t + \phi) = \frac{1}{2}m\omega^2 a^2[1 - \sin^2(\omega t + \phi)] = \frac{1}{2}m\omega^2(a^2 - x^2) \text{....(5)}$$

Therefore, the total energy of the particle at $x$ is,

$$E = T + V = \frac{1}{2}m\omega^2(a^2 - x^2) + \frac{1}{2}m\omega^2 x^2, \quad E = \frac{1}{2}m\omega^2 a^2 \qquad \text{....(6)}$$

The total energy $E$ is proportional to $a^2$. Classically, the particle can oscillate with any amplitude $a$ and as such the total energy increases with the increase of $a$.

The potential energy, $V = \frac{1}{2}kx^2$ has a parabolic form figure signifying that the particle executes motion in a potential well of parabolic form.

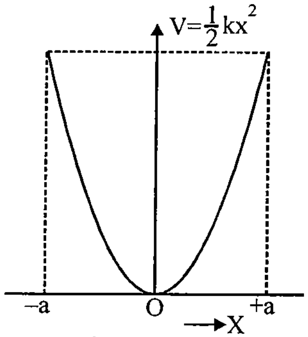


**Fig.** Potential energy curve of a LHO

**Classical Probability of Finding the LHO within a Small Distance *dx*:**

The probability of finding the LHO within $x$ and $x + dx$ from its equilibrium position is

$$= \frac{\text{time taken by the LHO to pass over dx}}{\text{period of oscillation}} \quad \text{or,} \quad P(x)dx = \frac{2dt}{T} = \frac{2dt}{\left(\frac{2\pi}{\omega}\right)} = \frac{\omega dt}{\pi}$$

But, $\because x = a\sin(\omega t + \phi)$, $\therefore dx = a\omega\cos(\omega t + \phi)dt$

$$\Rightarrow \quad dt = \frac{dx}{a\omega\sqrt{1 - x^2/a^2}} = \frac{dx}{\omega\sqrt{a^2 - x^2}} = \frac{dx}{\omega\sqrt{\frac{2E}{m\omega^2} - x^2}}, \quad \left[\because a^2 = \frac{2E}{m\omega^2}\right]$$

Therefore, $$P(x)dx = \frac{dx}{\pi\sqrt{\frac{2E}{m\omega^2} - x^2}} \quad \text{....(7)}$$

## 7.2. Quantum Mechanical Linear Harmonic Oscillator:

The Hamiltonian operator for the one-dimensional linear harmonic oscillator is given by,

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + V(x) = \frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2x^2 \quad \text{....(8)}$$

The time-independent Schrodinger wave equation for the LHO is,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi \quad \text{or,} \quad \frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi - \frac{2m}{\hbar^2}\left(\frac{1}{2}m\omega^2x^2\right)\psi = 0$$

$$\Rightarrow \quad \frac{d^2\psi}{dx^2} + \left(\frac{2mE}{\hbar^2}\right)\psi - \left(\frac{m^2\omega^2}{\hbar^2}\right)x^2\psi = 0 \quad \text{....(9)}$$

Equation may be expressed in a simpler form by introducing a dimensional independent variable $q$ such that,

$$q=\sqrt{\frac{m\omega}{\hbar}}x=\sqrt{\alpha}x,\ \left[\text{where, } \alpha=\frac{m\omega}{\hbar}\right] \quad \text{or, } x=\sqrt{\frac{\hbar}{m\omega}}q \qquad ....(10)$$

$$\therefore \quad \frac{d\psi}{dx}=\frac{d\psi}{dq}\frac{dq}{dx}=\frac{d\psi}{dq}\sqrt{\frac{m\omega}{\hbar}} \quad \text{or,} \quad \frac{d^2\psi}{dx^2}=\frac{d^2\psi}{dq^2}\frac{dq}{dx}\sqrt{\frac{m\omega}{\hbar}}=\frac{d^2\psi}{dq^2}\frac{m\omega}{\hbar} \quad \text{or,} \quad \frac{d^2\psi}{dx^2}=\frac{m\omega}{\hbar}\frac{d^2\psi}{dq^2}$$

Now from equations (9) we get,

$$\frac{m\omega}{\hbar}\frac{d^2\psi}{dq^2}+\left(\frac{2mE}{\hbar^2}-\frac{m^2\omega^2}{\hbar^2}\frac{\hbar}{m\omega}q^2\right)\psi=0 \quad \text{or,} \quad \frac{m\omega}{\hbar}\frac{d^2\psi}{dq^2}+\left(\frac{2mE}{\hbar^2}-\frac{m\omega}{\hbar}q^2\right)\psi=0$$

$$\Rightarrow \frac{d^2\psi}{dq^2}+\left(\frac{2E}{\hbar\omega}-q^2\right)\psi=0 \quad \text{or,} \quad \frac{d^2\psi}{dq^2}+(\lambda-q^2)\psi=0 \ \text{ where, } \lambda=\frac{2E}{\hbar\omega} \qquad ....(11)$$

**Asymptotic Solution:**

For large value of $q$ such that $q^2 \gg \lambda$, we may neglect $\lambda$ in equation (11), then it reduces to

$$\frac{d^2\psi}{dq^2}-q^2\psi=0 \qquad ....(12)$$

The solution of the above equation is, $\psi(q)=e^{\pm q^2/2}$ which may be verified as follows:

Differentiating equation twice with respect to $q$ we get,

$$\frac{d^2\psi}{dq^2}=q^2e^{\pm q^2/2}\pm e^{\pm q^2/2}=(q^2\pm 1)e^{\pm q^2/2}$$

Since $q$ is very large, $q^2\pm 1\simeq q^2 \quad \therefore \frac{d^2\psi}{dq^2}=q^2e^{\pm q^2/2}=q^2\psi$ or, $\frac{d^2\psi}{dq^2}-q^2\psi=0$

which is the same as equation (12).

Now, the probability of finding the LHO along the $x$-axis is $|\psi(q)|^2$ and for $\psi=e^{q^2/2}$, $|\psi(q)|^2 \to \infty$ as $q \to e^{q^2/2}$. Hence, $\psi(q) \mp \infty$ is not well behaved and as much this solution is rejected. On the other hand, $|\psi(q)|^2 \to 0$ as $q\to\infty$ in $\psi(q)=e^{-q^2/2}$. So $\psi(q)=e^{-q^2/2}$ is an asymptotic solution of the wave equation.

Let us now try with a possible solution, $\psi(q)=e^{-q^2/2}\phi(q)$ where, $\phi(q)$ is a function of $q$.

Differentiating equation twice with respect to $q$ we get,

$$\frac{d^2\psi}{dq^2}=e^{-q^2/2}\left[\frac{d^2\phi}{dq^2}-2q\frac{d\phi}{dq}+(q^2-1)\phi\right] \qquad ....(13)$$

Now from equations we get,

$$e^{-q^2/2}\left[\frac{d^2\phi}{dq^2}-2q\frac{d\phi}{dq}+(q^2-1)\phi\right]+e^{-q^2/2}(\lambda-q^2)\psi=0 \text{ or, } \quad e^{-q^2/2}\left[\frac{d^2\phi}{dq^2}-2q\frac{d\phi}{dq}+(\lambda-1)\phi\right]=0$$

But, $e^{-q^2/2}\neq 0$ except for $q\to\pm\infty$

So, $$\frac{d^2\phi}{dq^2}-2q\frac{d\phi}{dq}+(\lambda-1)\phi=0 \qquad \text{....(14)}$$

**Power Series Solution of Equation:**

The power series solution of equation may be assumed as,

$$\phi(q)=q^s\sum_{r=0}^{\infty}a_r q^r=\sum_r a_r q^{s+r} \qquad \text{....(15)}$$

Differentiating equation with respect to $q$ we get,

$$\frac{d\phi}{dq}=\sum_r a_r(s+r)q^{s+r-1} \text{ and } \frac{d^2\phi}{dq^2}=\sum_r a_r(s+r)(s+r-1)q^{s+r-2}$$

Using these values of $\frac{d\phi}{dq}$ and $\frac{d^2\phi}{dq^2}$ in equation we get,

$$\sum_r a_r(s+r)(s+r-1)q^{s+r-2}-2q\sum_r a_r(s+r)q^{s+r-1}+(\lambda-1)\sum_r a_r q^{s+r}=0$$

$$\Rightarrow \quad \sum_r a_r(s+r)(s+r-1)q^{s+r-2}-\sum_r a_r\{2(s+r)-(\lambda-1)\}q^{s+r}=0$$

The power series expressed by equation will be satisfied only if the coefficient of each power of $q$ vanishes separately.

Equating the coefficient of $q^{s-2}$ to zero, we get, $a_0 s(s-1)=0$, $\because a_0\neq 0, s=0$ or, $(s-1)=0$

Equating the coefficient of $q^{s+1}$ to zero, $a_1(s+1)s=0$

$\therefore$ either, $a_1=0$ or, $s=0$ or, $s=-1$. But, as $s\neq -1$

$\therefore$ either, $a_1=0$ or, $s=0$ or, both.

Similarly, equating the coefficient of $q^{s-r}$ to zero,

$$a_{r+2}(s+r+2)(s+r+1)-a_r[2(s+r)-(\lambda-1)]=0$$

$$\Rightarrow \quad a_{r+2}=\frac{2s+2r+1-\lambda}{(s+r+2)(s+r+1)}a_r \qquad \text{....(16)}$$

Equation is called the recursion formula. From this equation the coefficients of each even power of $q$, i.e. $q^2, q^4, q^6, \ldots$ can be calculated if $q_0$ is known. Similarly, the coefficients of $q^3, q^5, q^7, \ldots.$ can be found out if $q_1$ is known.

**Examining the Validity of the Series Solution:**

From equation we note that for large value of $r$, $\lim\limits_{r\to\infty}\dfrac{a_{r+2}}{a_r}=\dfrac{2}{r}$

Again, considering the series expansion,

$$e^{q^2}=1+q^2+\frac{q^4}{2!}+.....+\frac{q^n}{\left(\frac{r}{2}\right)!}+\frac{q^{r+2}}{\left(\frac{r}{2}+1\right)!}+....=1+b_2q^2+b_4q^4+....+b_rq^r+b_{r+2}r^{r+2}+....$$

The recursion formula for the expansion of $e^{q^2}$ is,

$$\frac{b_{r+2}}{b_r}=\frac{\left(\frac{r}{2}\right)!}{\left(\frac{r}{2}+1\right)!}=\frac{1}{\frac{r}{2}+1}=\frac{2}{r+2},\qquad \therefore\ \lim_{r\to\infty}\frac{b_{r+2}}{b_r}=\frac{2}{r}$$

It is, therefore, clear that for large value of $n$ the series for $\phi(q)$ and $e^{q^2}$ behave similarly. Then the corresponding eigenfunction is, $\psi(q)=e^{-q^2/2}\phi(q)=e^{-q^2/2}e^{q^2}=e^{q^2/2}$

Equation shows that for $q\to\infty$, $\psi(q)\to\infty$ and hence the wave function $\psi(q)$ is inadmissible as it does not satisfy the well behaved character of the function.

**Eigenvalues of the LHO:**

The requirement of the well behaved solution imposes the condition that the series represented by equation must terminate after the $r$th term. The condition for this is,

$$2s+2r+1-\lambda=0 \text{ or, } \lambda=2s+2r+1$$

$$\Rightarrow\quad \lambda=2(s+r)+1=2n+1\ [\text{where, } n=s+r \text{ and } n=0,1,2,.....]$$

$$\Rightarrow\quad \frac{2En}{\hbar\omega}=2n+1\left[\because \lambda=\frac{2E}{\hbar\omega}\right]\Rightarrow E_n=(2n+1)\frac{\hbar\omega}{2}$$

$$\Rightarrow\quad \boxed{E_n=\left(n+\tfrac{1}{2}\right)\hbar\omega} \qquad ....(17)$$

where, $\omega=2\pi\nu$, the angular frequency of the LHO and $n$ = 0, 1, 2, ....

The equation leads to the fact that there is a discrete set of energy eigenvalues which are equispaced in the interval of $\hbar\omega$.

For $n=0$, we get, $E_0=\dfrac{\hbar\omega}{2}=\dfrac{h\nu}{2}$, where $E_0$ is called the zero point (or, null point) energy of the oscillator.

**Wave Functions for the LHO:**

Putting $\lambda=2n+1$ in equation we get, $\dfrac{d^2\phi}{dq^2}-2q\dfrac{d\phi}{dq}+2nq=0$ ....(18)

This is known as the Hermite differential equation. We can prove that the solution of equation is of the form,

$$\phi(q) = e^{q^2}\left(\frac{d}{dq}\right)^n e^{-q^2} \qquad \text{....(19)}$$

Multiplying equation by $(-1)^n$ we get Hermite polynomial of order $n$,

$$H_n(q) = (-1)\phi(q) = (-1)^n e^{q^2}\left(\frac{d}{dq}\right)^n e^{-q^2} \qquad \text{....(20)}$$

The following are a few Hermite polynomials for even and odd values of $n$:

$n = 0,\ H_0(q) = 1$ $\qquad$ $n = 1,\ H_1(q) = 2q$

$n = 2,\ H_2(q) = 4q^2 - 2$ $\qquad$ $n = 3,\ H_3(q) = 8q^3 - 12q$

$n = 4,\ H_4(q) = 16q^4 - 48q^2 + 12$ $\qquad$ $n = 5,\ H_5(q) = 32q^5 - 160q^3 + 120q$

Now the eigenfunction for the LHO is given by, $\psi_n(q) = N_n e^{-q^2/2} H_n(q)$ ....(21)

Here, $N_n$ is the normalization constant. $N_n$ can be found out from the condition of normalization as follows:

$$\int_{-\infty}^{+\infty} |\psi_n(x)|^2 dx = 1 \text{ or, } \frac{|N_n|^2}{\sqrt{\alpha}}\int_{-\infty}^{+\infty} H_n^2(q) e^{-q^2} dq = 1$$

But, $\displaystyle\int_{-\infty}^{+\infty} H_n^2(q) e^{-q^2/2} dq = 2^n (n!)\sqrt{\pi}$; So, $\displaystyle\frac{N_n^2}{\sqrt{\alpha}}.2^n (n!)\sqrt{\pi} = 1$

$$\Rightarrow \quad N_n = \left[\frac{\sqrt{\alpha}}{2^n n!\sqrt{\pi}}\right]^{1/2} \qquad \text{....(22)}$$

Thus, the normalized wave function for the LHO is,

$$\psi_n(x) = \left[\frac{\sqrt{\alpha}}{2^n n!\sqrt{\pi}}\right]^{1/2} e^{-\alpha x^2/2} H_n\left(\sqrt{\alpha}\,x\right) \qquad \text{....(23)}$$

$$\Rightarrow \quad \psi_n(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} (2^n n!)^{-1/2} e^{-\alpha x^2/2} H_n\left(\sqrt{\alpha}\,x\right) \qquad \text{....(24)}$$

This is the normalized eigenfunction of the oscillator in the $n$th state. The wave function corresponding to the lowest energy state, i.e., $n = 0$ is,

$$\psi_0(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2} \quad [\because H_0(\sqrt{\alpha}\,x) = 1] \qquad \text{....(25)}$$

- The table below shows first few wave functions of the oscillator:

**Table : First few wave function of the LHO**

| $n$ | $\psi_n(x)$ |
|---|---|
| 0 | $\psi_0(x) = (\alpha/\pi)^{1/4} e^{-\alpha x^2/2}$ |
| 1 | $\psi_1(x) = (\alpha/\pi)^{1/4}\sqrt{2}\sqrt{\alpha}\, x e^{-\alpha x^2/2}$ |
| 2 | $\psi_2(x) = (\alpha/\pi)^{1/4}(1/\sqrt{2})(2\alpha x^2 - 1)e^{-\alpha x^2/2}$ |
| 3 | $\psi_3(x) = (\alpha/\pi)^{1/4}(1/\sqrt{3})[2(\alpha\sqrt{x})^3 - 2\sqrt{\alpha}\, x]e^{-\alpha x^2/2}$ |
| 4 | $\psi_4(x) = 1/2(\alpha/\pi)^{1/4}(1/\sqrt{6})[4(\sqrt{\alpha}\, x)^4 - 12(\sqrt{\alpha}\, x)^2 + 3]e^{-\alpha x^2/2}$ |
| 5 | $\psi_5(x) = 1/2(\alpha/\pi)^{1/4}(1/\sqrt{15})[4(\sqrt{\alpha}\, x)^5 - 20(\sqrt{\alpha}\, x)^3 + 15(\sqrt{\alpha x})]e^{-\alpha x^2/2}$ |

It is to be noted that $\psi_0(x)$ has no node for finite $q$. As $n$ increases, the number of nodes progressively increases with $n$. For example, $\psi_1(x)$ has a single node and $\psi_2(0)$ has two nodes as shown in figure.

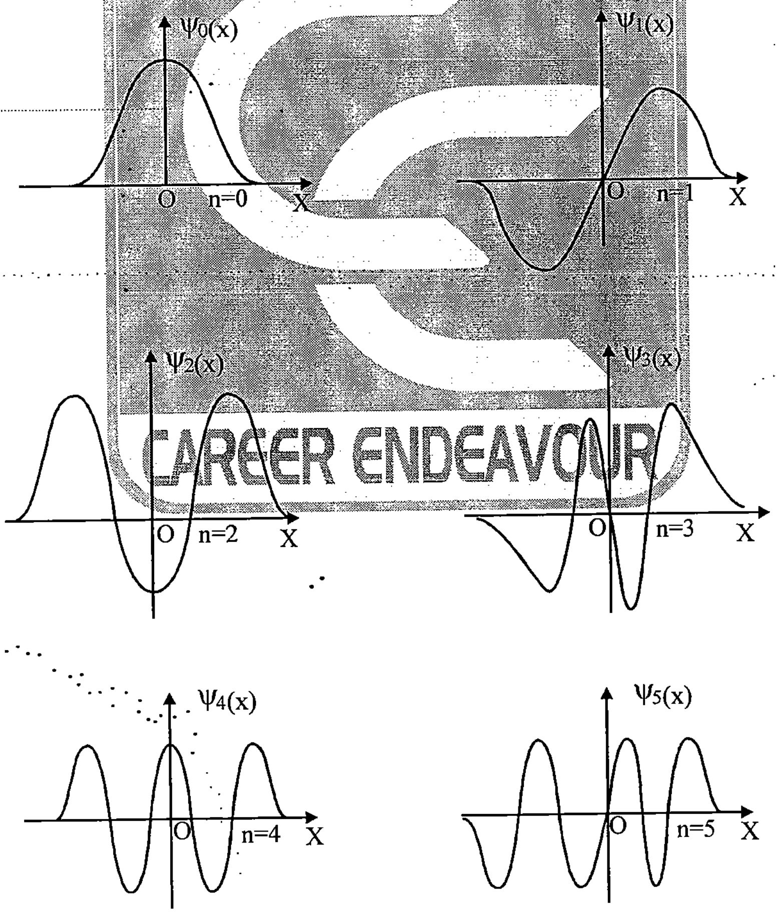


**Fig.** Harmonic oscillator wave functions for different $n$

**Operator formalism of harmonic Oscillator:**

Hamiltonian of the system is $\hat{H}=\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2$

Defining two dimensionless operators

$$\hat{P}=\frac{\hat{p}}{\sqrt{m\hbar\omega}} \text{ and } \hat{X}=\hat{x}\sqrt{\frac{m\omega}{\hbar}}$$

The hamiltonian can be written as, $\hat{H}=\frac{\hbar\omega}{2}\left[\frac{\hat{p}^2}{m\hbar\omega}+\frac{m\omega}{2\hbar}\hat{x}^2\right]=\frac{\hbar\omega}{2}\left[\hat{P}^2+\hat{X}^2\right]$

Introducing two non-hermitian operators, $\hat{a}=\frac{1}{\sqrt{2}}\left[\hat{X}+i\hat{P}\right]$, $\hat{a}^\dagger=\frac{1}{\sqrt{2}}\left[\hat{X}-i\hat{P}\right]$

where $\hat{a}$ and $\hat{a}^\dagger$ are hermitian conjugate to each other.

Now, $\hat{a}^\dagger\hat{a}=\frac{1}{2}\left[\hat{X}^2+\hat{P}^2+i\hat{X}\hat{P}-i\hat{P}\hat{X}\right]=\frac{1}{2}\left(\hat{P}^2+\hat{X}^2\right)+\frac{i}{2}\left[\hat{X},\hat{P}\right]$

$\Rightarrow \hat{a}^\dagger\hat{a}=\frac{\hat{H}}{\hbar\omega}-\frac{1}{2} \Rightarrow \hat{H}=\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)\hbar\omega \quad \Rightarrow \hat{H}=\left(\hat{N}+\frac{1}{2}\right)\hbar\omega$

where, $\hat{N}=\hat{a}^\dagger\hat{a}$ = number operator/occupation number operator.

- **Some important commutation relations:**

(i) $\left[\hat{a},\hat{a}^\dagger\right]=1$ (ii) $\left[\hat{a}^\dagger\hat{a},\hat{a}^\dagger\right]=\hat{a}^\dagger$ (iii) $\left[\hat{a}^\dagger\hat{a},\hat{a}\right]=-\hat{a}$

(iv) $\left[\hat{H},\hat{a}^\dagger\hat{a}\right]=0$ (v) $\left[\hat{a},\hat{H}\right]=\hbar\omega\hat{a}$ (vi) $\left[\hat{a}^\dagger,\hat{H}\right]=-\hbar\omega\hat{a}^\dagger$

- **Lowering operator and Raising operator:**

$\hat{H}\left(\hat{a}|n\rangle\right)=\left(\hat{a}\hat{H}-\hbar\omega\hat{a}\right)|n\rangle=\hat{a}E_n|n\rangle-\hbar\omega\hat{a}|n\rangle=\left(E_n-\hbar\omega\right)\left(\hat{a}|n\rangle\right)$

So, that, $\hat{a}|n\rangle$ is the eigenstates of $\hat{H}$ operator corresponding eigenvalue $\left(E_n-\hbar\omega\right)$. Action of $\hat{a}$ on $|n\rangle$ generate new energy states that is lower by one unit energy $\hbar\omega$ i.e.

$$\hat{a}|n\rangle=C_n|n-1\rangle \quad \Rightarrow \langle n|\hat{a}^\dagger=\langle n-1|c_n^*$$

Therefore, $\left\langle n\middle|\hat{a}^\dagger\hat{a}\middle|n\right\rangle=\langle n-1|n-1\rangle|c_n|^2 \Rightarrow n=|c_n|^2 \Rightarrow |c_n|=\sqrt{n}$

So, $\quad \hat{a}|n\rangle=\sqrt{n}|n-1\rangle$

Similarly, $\hat{H}\left(\hat{a}^\dagger|n\rangle\right)=\left(\hat{a}^\dagger\hat{H}+\hbar\omega\hat{a}^\dagger\right)|n\rangle=\left(E_n+\hbar\omega\right)\left(\hat{a}^\dagger|n\rangle\right)$

So, that, $\hat{a}^\dagger|n\rangle$ are the eigenstates of $\hat{H}$ operator corresponding eigenvalue $\left(E_n+\hbar\omega\right)$. Action of $\hat{a}^\dagger$ on $|n\rangle$ generate new energy states that is higher by one unit energy $\hbar\omega$.

$$\hat{a}^\dagger|n\rangle=c_n|n+1\rangle \quad \Rightarrow \langle n|\hat{a}=\langle n+1|c_n^*$$

Therefore, $\left\langle n\left|\hat{a}\hat{a}^{\dagger}\right|n\right\rangle=\left|c_n\right|^2\langle n+1\mid n+1\rangle\Rightarrow(n+1)=\left|c_n\right|^2\Rightarrow\left|c_n\right|=\sqrt{n+1}$

So, $$\hat{a}^{\dagger}|n\rangle=\sqrt{n+1}|n+1\rangle$$

Therefore, the nth state can be written in terms of ground state as

$$|n\rangle=\frac{\left(\hat{a}^{\dagger}\right)^n}{\sqrt{n!}}|0\rangle$$

**Calculation of normalized wave function:**

**Ground state wavefunction:**

Since, action of $\hat{a}$ on $|n\rangle$ generate new energy states that is lower by one unit energy $\hbar\omega$, then

$$\hat{a}|0\rangle=0$$

$$\Rightarrow\left\{\sqrt{\frac{m\omega}{2\hbar}}\hat{x}+i\frac{\hat{p}}{\sqrt{2m\omega\hbar}}\right\}\psi_0=0\Rightarrow\sqrt{\frac{m\omega}{2\hbar}}x\psi_0+\frac{\hbar}{\sqrt{2m\omega\hbar}}\psi_0=0$$

$$\Rightarrow\frac{d\psi_0}{dx}=-\frac{m\omega}{\hbar}x\psi_0\Rightarrow\frac{d\psi_0}{\psi_0}=-\frac{m\omega}{\hbar}xdx\Rightarrow\ell n\psi_0=-\frac{m\omega}{\hbar}\frac{x^2}{2}+\ell nA$$

$$\Rightarrow\psi_0(x)=A\exp\left(-\frac{m\omega}{\hbar}\frac{x^2}{2}\right)\Rightarrow\psi_0(x)=A\exp\left(-\frac{\alpha^2x^2}{2}\right)\ \left[\text{where } \alpha^2=\frac{m\omega}{\hbar}\right]$$

Normalization Condition: $\int_{-\infty}^{\infty}\psi_0^*(x)\psi_0(x)dx=1$

$$\Rightarrow 2|A|^2\int_0^{\infty}\exp\left(-\alpha^2x^2\right)dx=1\Rightarrow A=\left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2}$$

Therefore, normalized ground state wavefunction will be

$$\psi_0(x)=\left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2}e^{-\alpha^2x^2/2}$$

**First excited state wavefunction:**

Since, $\hat{a}^{\dagger}|0\rangle=|1\rangle$

$$\Rightarrow\quad|1\rangle=\left[\sqrt{\frac{m\omega}{2\hbar}}\hat{x}-\frac{i}{\sqrt{2m\omega\hbar}}\hat{p}\right]|0\rangle$$

$$=\left[\sqrt{\frac{m\omega}{2\hbar}}\hat{x}-\frac{i}{\sqrt{2m\omega\hbar}}\left(i\hbar\frac{\partial}{\partial x}\right)\right]|0\rangle$$

$$=\sqrt{\frac{m\omega}{2\hbar}}x\left[\frac{\alpha}{\sqrt{\pi}}\right]^{1/2}e^{\frac{-\alpha^2x^2}{2}}-\frac{\hbar}{\sqrt{2m\omega\hbar}}\left[\frac{\alpha}{\sqrt{\pi}}\right]^{1/2}e^{-\alpha^2x^2/2}\left(-\alpha^2x\right)$$

$$=\left[\frac{\alpha}{\sqrt{4}.\sqrt{\pi}}\right]^{1/2}e^{-\frac{\alpha^2x^2}{2}}(2\alpha x)=\left[\frac{\alpha}{2\sqrt{\pi}}\right]^{1/2}e^{-\frac{\alpha^2x^2}{2}}H_1(\alpha x)$$

**Expectation value of a odd operator for a symmetric potential:**

The wave function of a particle is moving under a symmetric potential $V(-x)=V(x)$, has a definite parity i.e it will be of either even or odd parity.

Therefore, expectation value of an odd operator $=\langle\hat{A}\rangle=\langle\psi|\hat{A}|\psi\rangle=0$

Similarly, $\langle\hat{A}^m\rangle=\langle\psi|\hat{A}^m|\psi\rangle=0\ (m=\text{odd integer})$

**Calculation of expectation value using operator formalism:**

As we know, $\hat{a}=\sqrt{\frac{m\omega}{2\hbar}}\hat{x}+\hat{p}\frac{i}{\sqrt{2m\omega\hbar}}$ and $\hat{a}^\dagger=\sqrt{\frac{m\omega}{2\hbar}}\hat{x}-\hat{p}\frac{i}{\sqrt{2m\omega\hbar}}$,

therefore we get,

$$\hat{x}=\sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}+\hat{a}^\dagger\right) \quad \text{and} \quad \hat{p}=\frac{1}{i}\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a}-\hat{a}^\dagger\right)$$

$$\langle\hat{x}\rangle=\sqrt{\frac{\hbar}{2m\omega}}\left\langle n\left|\left(\hat{a}+\hat{a}^\dagger\right)\right|n\right\rangle=\sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\langle n|n-1\rangle+\sqrt{n+1}\langle n|n+1\rangle\right]=0$$

$$\langle\hat{p}\rangle=\frac{1}{i}\sqrt{\frac{\hbar m\omega}{2}}\left\langle n\left|\hat{a}-\hat{a}^\dagger\right|n\right\rangle=\frac{1}{i}\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n}\langle n|n-1\rangle-\sqrt{n+1}\langle n|n+1\rangle\right]=0$$

$$\langle\hat{x}^2\rangle=\frac{\hbar}{2m\omega}\left\langle n\left|\left(\hat{a}+\hat{a}^\dagger\right)\left(\hat{a}^\dagger+\hat{a}\right)\right|n\right\rangle=\frac{\hbar}{2m\omega}\left\langle n\left|\left(\hat{a}^2+\hat{a}^{\dagger^2}+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}\right)\right|n\right\rangle$$

$$=\frac{\hbar}{2m\omega}\left[\sqrt{n}\sqrt{n-1}\langle n|n-2\rangle+\sqrt{n+1}\sqrt{n+2}\langle n|n+1\rangle+\sqrt{n+1}\sqrt{n+1}\langle n|n\rangle+\sqrt{n}\sqrt{n}\langle n|n\rangle\right]$$

$$=\frac{\hbar}{2m\omega}(2n+1)=\frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)$$

Similarly, $\langle p^2\rangle=\left(n+\frac{1}{2}\right)m\hbar\omega$

Uncertainity in position $\Delta x=\sqrt{\langle x^2\rangle-\langle x\rangle^2}=\sqrt{\frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)}$

Uncertainity in momentum $\Delta p=\sqrt{\langle p^2\rangle-\langle p\rangle^2}=\sqrt{\left(n+\frac{1}{2}\right)\hbar m\omega}$

Therefore, the uncertainity product becomes $\Delta x\Delta p=\left(n+\frac{1}{2}\right)\hbar$

**Example 1:** Let $|0\rangle$ and $|1\rangle$ denote the normalized eigenstates corresponding to the ground and first state of a 1-D harmonic oscillator. The uncertainty $\Delta p$ in the state $\frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)$ is:

(a) $\frac{\sqrt{m\omega\hbar}}{2}$ (b) $\sqrt{\frac{m\omega\hbar}{2}}$ (c) $\sqrt{m\omega\hbar}$ (d) $\sqrt{2m\omega\hbar}$

**Soln.** $\langle \hat{p} \rangle = \frac{1}{2}\left[\langle 0|\hat{p}|0\rangle + \langle 0|\hat{p}|1\rangle + \langle 1|\hat{p}|0\rangle + \langle 1|\hat{p}|1\rangle\right] = \frac{1}{2}\left[0 + \frac{1}{i}\sqrt{\frac{m\omega\hbar}{2}} - \frac{1}{i}\sqrt{\frac{m\omega\hbar}{2}} + 0\right] = 0$

$$\langle \hat{p}^2 \rangle = \frac{1}{2}\left[\langle 0|\hat{p}^2|0\rangle + \langle 0|\hat{p}^2|1\rangle + \langle 1|\hat{p}^2|0\rangle + \langle 1|\hat{p}^2|1\rangle\right] = \frac{1}{2}\left[m\hbar\omega\frac{1}{2} + 0 + 0 + \frac{3}{2}m\hbar\omega\right] = m\omega\hbar$$

Uncertainity in the momentum in the given state will be

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\hbar m\omega}$$

**Example 2:** The wave function of a linear harmonic oscillator in the potential $V(x) = \frac{1}{2}m\omega^2 x^2$ at a given instant is $\psi(x) = \left(\frac{2}{\pi\beta^2}\right)^{1/4} e^{-x^2/\beta^2}$. For what value of $\beta$ will the function $a\psi(x)$ be zero?

**Soln.** $a = \frac{1}{\sqrt{2}}\left(\hat{X} + i\hat{P}_x\right) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}x + i\frac{p_x}{\sqrt{m\hbar\omega}}\right) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{m\omega}{\hbar}}x + \sqrt{\frac{\hbar}{m\omega}}\frac{d}{dx}\right)$

Thus, $a\psi(x) = \frac{1}{\sqrt{2}}\left(\frac{2}{\pi\beta^2}\right)^{1/4}\left(\sqrt{\frac{m\omega}{\hbar}}x + \sqrt{\frac{\hbar}{m\omega}}\frac{d}{dx}\right)e^{-x^2/\beta^2}$

$$= \left(\frac{1}{2\pi\beta^2}\right)^{1/4}\left(\sqrt{\frac{m\omega}{\hbar}}x + \sqrt{\frac{\hbar}{m\omega}}\left(-\frac{2x}{\beta^2}\right)\right)e^{-x^2/\beta^2}$$

$$= \left(\frac{1}{2\pi\beta^2}\right)^{1/4}\left(\sqrt{\frac{m\omega}{\hbar}} - \frac{2}{\beta^2}\sqrt{\frac{\hbar}{m\omega}}\right)xe^{-x^2/\beta^2}$$

This will be zero if $\beta = \sqrt{\frac{2\hbar}{m\omega}}$.

**Example 3.** Consider a particle of mass 'm' and charge 'q' placed in a uniform electric field $E = E_0\hat{i}$. Apart from the force due to the electric field, a restoring force corresponding to the potential $V_1(x) = \frac{1}{2}m\omega^2 x^2$ acts on the particle. Find the energy eigenvalues of the particle, taking the electric potential energy to be zero at the origin.

**Soln.** The electric potential enregy at the position 'x' will be $-qE_0 x$. Thus, the net potential is

$$V(x) = \frac{1}{2}m\omega^2 x^2 - qE_0 x = \frac{1}{2}m\omega^2\left(x^2 - \frac{2qE_0}{m\omega^2}x\right)$$

$$= \frac{1}{2}m\omega^2\left(x - \frac{qE_0}{m\omega^2}\right)^2 - \frac{q^2E_0^2}{2m\omega^2}$$

If you shift the origin of '$x$' to $\frac{qE_0}{m\omega^2}$ and define $x' = \left(x - \frac{qE_0}{m\omega^2}\right)$, the potential as a function of $x'$ is

$$V(x') = \frac{1}{2}m\omega^2 x'^2 - \frac{q^2E_0^2}{2m\omega^2}$$

This is the potential of a linear harmonic oscillator with energy $-\frac{q^2E_0^2}{2m\omega^2}$ at $x' = 0$. This is a constant shift in the energy eigenvalues are $\left(n + \frac{1}{2}\right)\hbar\omega - \frac{q^2E_0^2}{2m\omega^2}$.

**Example 4.** A linear harmonic oscillator is in its ground state. Find the range of '$x$' that is allowed classically. Calculate the probability of finding it in the classically forbidden region.

**Soln.** For classically allowed region, the total energy E should be greater than or equal to the potential energy. For linear harmonic oscillator in ground state, this region is given by

$$\frac{\hbar\omega}{2} \ge \frac{1}{2}m\omega^2 x^2 \Rightarrow x^2 \le \frac{\hbar}{m\omega} \Rightarrow -\sqrt{\frac{\hbar}{m\omega}} \le x\sqrt{\frac{\hbar}{m\omega}}$$

The probability of finding the oscillator in this range is

$$\int_{-\sqrt{\hbar/m\omega}}^{\sqrt{\hbar/m\omega}} \left|\psi_0(x)^2\right|^2 dx = 2 \times \int_0^{\sqrt{\hbar/m\omega}} \sqrt{\frac{m\omega}{\pi\hbar}} e^{-\left(\frac{m\omega}{\hbar}\right)x^2} dx$$

Assuming, $t = \sqrt{\frac{m\omega}{\hbar}}x$, the above expression becomes $2\sqrt{\frac{m\omega}{\pi\hbar}}\sqrt{\frac{\hbar}{m\omega}}\int_0^1 e^{-t^2}dt \approx 0.84$

Thus the probability of finding the oscillator in the classically forbidden region is $1 - 0.84 = 0.16$.

**Example 5.** A particle of mass 'm' moves in the potential $V(x) = \frac{1}{2}m\omega^2 x^2$. It has wave function $\psi(x)$ at time t = 0. What will be the wave function at $t = \frac{2\pi}{\omega}$ (one time period)?

**Soln.** Let the wave function at $t = 0$ be $\psi(x) = \sum_n C_n |\phi_n\rangle$

The wave function at time $t$ will be

$$\psi(x,t) = \sum_n C_n |\phi_n\rangle e^{-\frac{i}{\hbar}E_n t} = \sum_n C_n |\phi_n\rangle e^{-\frac{i}{\hbar}\left(n+\frac{1}{2}\right)\hbar\omega t}$$

$$= \sum_n C_n |\phi_n\rangle e^{-i\left(n+\frac{1}{2}\right)2\pi} = \sum_n C_n |\phi_n\rangle e^{-i\pi} = -\sum_n C_n |\phi_n\rangle = -\psi(x)$$

In one time period the wave function becomes negative of its original form. In two time periods, it regains its original shape.

**Example 6.** A linear harmonic oscillator is in the first excited state. (i) At what point is its probability density maximum? (ii) What is the value of maximum probability density?

**oln.** The harmonic oscillator wave function in the $n = 1$ state is

$$\psi_1(x) = \left(\frac{\alpha}{2\sqrt{\pi}}\right)^{1/2} 2\alpha x \exp\left(\frac{-\alpha^2 x^2}{2}\right) \quad \text{where } \alpha = \left(\frac{m\omega}{\hbar}\right)^{1/2}$$

(i) Probability density $P(x) = \psi\psi^* = \frac{2\alpha^3}{\sqrt{\pi}} x^2 \exp(-\alpha^2 x^2)$

$P(x)$ is maximum when $dP/dx = 0 \Rightarrow \frac{2\alpha^3}{\sqrt{\pi}}(2x - 2\alpha^2 x^3) = 0 \;\Rightarrow x = \pm\frac{1}{\alpha}$

(ii) Maximum value of $P(x) = \frac{2\alpha}{\sqrt{\pi}}\frac{1}{e} = \frac{2\alpha}{\sqrt{\pi}}\frac{1}{2.718} = 0.415\alpha$

**Example** 7. The energy of a linear harmonic oscillator in third excited state is 4.1 eV. The frequency of vibration is ($\mu = 6.62\times10^{-34}$ J-sec) **[BHU 2011]**

(a) $3.3\times10^{13}\ Hz$ (b) $2.8\times10^{13}\ Hz$ (c) $3.9\times10^{13}\ Hz$ (d) $4.2\times10^{13}\ Hz$

**Soln.** Given: Third excited state energy = $E_3 = \frac{7}{2}h\nu = 4.1eV$

Therefore, Frequency of vibration will be $\nu = \frac{2}{7}\frac{E_3}{h} = \frac{2}{7}\times\frac{4.1\times1.6\times10^{-19}}{6.6\times10^{-34}} = 0.28\times10^{15}\ Hz$

**Correct option is ( )**

**Example 8.** Let $|n\rangle$ represent the normalized $n^{\text{th}}$ energy eigenstate of the one-dimensional harmonic oscillator

$$H|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle$$

If $|\psi\rangle$ is a normalized ensemble state that can be expanded as a linear combination **[BHU 2012]**

$$|\psi\rangle = \frac{1}{\sqrt{14}}|1\rangle - \frac{2}{\sqrt{14}}|2\rangle + \frac{3}{\sqrt{14}}|3\rangle$$

of the eigenstates, what is the expectation value of the energy operator in this ensemble state ?

(a) $\frac{102}{14}\hbar\omega$ (b) $\frac{43}{14}\hbar\omega$ (c) $\frac{23}{14}\hbar\omega$ (d) $\frac{17}{14}\hbar\omega$

**Soln.** Given : $H|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle$ and $|\psi\rangle = \frac{1}{\sqrt{4}}|1\rangle - \frac{2}{\sqrt{14}}|2\rangle + \frac{3}{\sqrt{14}}|3\rangle$

Therefore, $\langle H\rangle = \langle n|H|n\rangle$

$$= \left(\frac{1}{\sqrt{14}}\langle 1| - \frac{2}{\sqrt{14}}\langle 2| + \frac{3}{\sqrt{14}}\langle 3|\right) H \left(\frac{1}{\sqrt{14}}|1\rangle - \frac{2}{\sqrt{14}}|2\rangle + \frac{3}{\sqrt{14}}|3\rangle\right)$$

$$= \left(\frac{1}{\sqrt{14}}\langle 1| - \frac{2}{\sqrt{14}}\langle 2| + \frac{3}{\sqrt{14}}\langle 3|\right)\left(\frac{3}{2}\frac{\hbar\omega}{\sqrt{14}}|1\rangle - \frac{5}{2}\hbar\omega\frac{2}{\sqrt{14}}|2\rangle + \frac{7}{2}\hbar\omega\frac{3}{\sqrt{14}}|3\rangle\right)$$

$$= \left(\frac{3}{2\times14} + \frac{10}{14} + \frac{63}{2\times14}\right)\hbar\omega = \left(\frac{3+20+63}{28}\right)\hbar\omega = \frac{43}{14}\hbar\omega.$$

**Correct option is (b)**

**Example 9.** The wave function $\psi(x) = A\exp\left\{-\frac{b^2x^2}{2}\right\}$ where $A$ and $b$ are real constants, is a normalized eigenfunction of the Schrödinger equation for a particle of mass $M$ and energy $E$ in a one-dimensional potential $V(x)$ such that $V(x) = 0$ at $x = 0$. Which of the following is correct? **[BHU 2012]**

(a) $V = \frac{\hbar^2 b^4}{2M}$ (b) $V = \frac{\hbar^2 b^4 x^2}{2M}$ (c) $V = \frac{\hbar^2 b^6 x^4}{2M}$ (d) $E = \hbar^2 b^2(1 - b^2x^2)$

**Soln.** Given : $\psi(x) = A\exp\left\{-\frac{b^2x^2}{2}\right\}$

The Schrödinger equation is $-\frac{\hbar^2}{2M}\frac{d^2}{dx}\psi(x) + V\,\psi(x) = E\,\psi(x)$

$$\Rightarrow -\frac{\hbar^2}{2M}\frac{d}{dx}\left\{-b^2x\,A\exp\left(-\frac{b^2x^2}{2}\right)\right\} + V\,\psi(x) = E\,\psi(x)$$

$$\Rightarrow \frac{\hbar^2b^2}{2M}\left\{1 - b^2x^2\right\}\psi(x) + V\,\psi(x) = E\,\psi(x)$$

$$\Rightarrow E = \frac{\hbar^2b^2}{2M}\left(1 - b^2x^2\right) + V$$

$$\Rightarrow V = E - \frac{\hbar^2b^2}{2M}\left(1 - b^2x^2\right)$$

$\because V(x) = 0$ at $x = 0$, we have $0 = E - \frac{\hbar^2b^2}{2M}$ or $E = \frac{\hbar^2b^2}{2M}$

$$\therefore V = \frac{\hbar^2b^4x^2}{2M}.$$

**Correct option is (b)**

**Example 10.** The ground state eigenfunction for a linear harmonic oscillator, in terms of $\alpha = \sqrt{\frac{mk}{\hbar^2}}$ where $k$= force constant and $m$ = mass of the linear oscillator, is **[BHU 2013]**

(a) $\psi(x) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/4} e^{\alpha x^2/2}$ (b) $\psi(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2}$

(c) $\psi(x) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/2} e^{-\alpha^2x^2/2}$ (d) $\psi(x) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{1/4} e^{+\alpha^2x^2/2}$

**Soln.** The ground state eigen function is

$$\psi(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\frac{\alpha x^2}{2}}$$

**Correct option is (b)**

**Example 11.** The ground state energy of the system described by the Hamiltonian $H=\frac{bx^2}{2m}+ax^2+bx+c$ is:

**[BHU 2014]**

(a) $\hbar\sqrt{\frac{2a}{m}}+c-\frac{b^2}{4a}$ (b) $\hbar\sqrt{\frac{2a}{m}}-c+\frac{b^2}{4a}$ (c) $\hbar\sqrt{\frac{a}{2m}}-c+\frac{b^2}{4a}$ (d) $\hbar\sqrt{\frac{a}{2m}}+c-\frac{b^2}{4a}$

**Soln.** The Hamiltonian is

$$H=ax^2+bx+c \qquad =a\left(x+\frac{b}{2}\right)^2-\frac{b^2}{4a}+c$$

This is the potential of simple harmonic oscillator with a shift of $-\frac{b^2}{4a}+c$.

On comparing with standard harmonic oscillator potential, $H=\frac{1}{2}kx^2+H_0$

we have $k=2a$ and $H_0=-\frac{b^2}{4a}+c$

Since, the minimum energy is $E=\frac{\hbar}{2}\sqrt{\frac{k}{m}}+H_0=\hbar\sqrt{\frac{a}{2m}}-\frac{b^2}{4a}+c$

**Correct option is (c)**

**Example 12.** Consider one sided simple harmonic oscillator described by the potential $V(x)=\frac{1}{2}mw^2x^2,\ 0\le x\le\infty$. The value of $\langle x\rangle$ in any state will be **[BHU 2014]**

(a) 0 (b) a positive number (c) always be negative (d) can be positive or negative

**Soln.** Given : $V(x)=\frac{1}{2}m\omega^2x^2,\ 0\le x\le\infty$

The wave function for this potential is

$$\psi(x)=\sqrt{\frac{\gamma}{2^n n!\sqrt{\pi}}}\,H_n(\xi)\,e^{-\xi^2/2},\ n=1,3,5,\ldots \left(\text{where } \xi=\gamma x \text{ and } \gamma=\sqrt{\frac{m\omega}{\hbar}}\right)$$

$$\therefore\ \langle x\rangle=\int_0^\infty x|\psi(x)|^2\,dx \qquad =\frac{\gamma}{2^n n!\sqrt{\pi}}\int_0^\infty x\left(H_n(\xi)\,e^{-\xi^2/2}\right)^2 dx$$

Taking $n=1$, we have $\langle x\rangle=\frac{\gamma}{2\sqrt{\pi}}\int_0^\infty x\,H_1^2(\xi)\,e^{-\xi^2}dx$

where $H_1(\xi)=2\xi$

$$\therefore\ \langle x\rangle=\frac{4\gamma}{2\sqrt{\pi}}\int_0^\infty x\,\xi^2\,e^{-\xi^2}dx \qquad =\frac{4\gamma^3}{2\sqrt{\pi}}\int_0^\infty x^3\,e^{-\gamma^2x^2}dx$$

Using, $\int_0^\infty x^n e^{-ax^2}dx=\frac{\Gamma\left(\frac{n+1}{2}\right)}{2(a)^{\frac{n+1}{2}}}$

$$\therefore\ \langle x\rangle=\frac{2\gamma^3}{\sqrt{\pi}}\,\frac{\Gamma(2)}{2(\gamma^2)^2}=\frac{1}{\gamma\sqrt{\pi}}=\sqrt{\frac{\hbar}{m\omega\pi}}>0$$

**Correct option is (b)**

**Example 13.** The wave function for a 1-d simple harmonic oscillator has odd number of real zeros, Which of the following statements is incorrect? **[BHU 2014]**

(a) The wave function is odd under parity

(b) The wave function passes through origin

(c) The wave function does not have definite parity

(d) The state for this wave function is non-degenerate

**Soln.** Odd number of zeroes means the wave function has odd parity. Therefore, the wavefunction will pass through the origin and the corresponding state will be non-degenerate.

**Correct option is (c)**

**Example 14.** State of a one-dimensional simple harmonic oscillators is $\psi(x,t)=\frac{1}{5}\left[3\phi_0-2\sqrt{2}\,\phi_1+2\sqrt{2}\,\phi_2\right]$, where $\phi_n$ are the eigen functions of the Hamiltonian with eigen values $E_n=\left(n+\frac{1}{2}\right)\hbar\omega$. The expectation value of the energy in the state $\psi(x,t)$ is : **[BHU 2014]**

(a) $0.46\,\hbar\omega$ (b) $1.46\,\hbar\omega$ (c) $0.5\,\hbar\omega$ (d) 0

**Soln.** Given : $\psi(x,t)=\frac{1}{5}\left[3\,\phi_0-2\sqrt{2}\,\phi_1+2\sqrt{2}\,\phi_2\right]$

$$\langle E\rangle=\left(\frac{3}{5}\right)^2E_0+\left(-\frac{2\sqrt{2}}{5}\right)^2E_1+\left(\frac{2\sqrt{2}}{5}\right)^2E_2$$

$$=\frac{9}{25}\frac{\hbar\omega}{2}+\frac{8}{25}\frac{3}{2}\hbar\omega+\frac{8}{25}\frac{5}{2}\hbar\omega=\frac{73}{50}\hbar\omega=1.46\,\hbar\omega$$

**Correct option is (b)**

**Example 15.** For a linear harmonic oscillator, the potential is $V=\frac{1}{2}m\omega^2x^2$. Using Schrödinger equation with $\psi(x)=xe^{-m\omega x^2/2\hbar}$ one obtains the energy eigenvalue as **[BHU 2015]**

(a) $E=\frac{5}{2}\hbar\omega$ (b) $E=\frac{3}{2}\hbar\omega$ (c) $E=\frac{1}{2}\hbar\omega$ (d) $E=\hbar\omega$

**Soln.** The wavefunction for a linear harmonic oscillator is

$$\psi_n(x)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(\xi)e^{-\xi^2/2},\text{ where }\xi=\sqrt{\left(\frac{m\omega}{\hbar}\right)}x$$

For $n=1$, $\psi_1(x)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2}}H_1(\xi)e^{-\xi^2/2}$, where $H_1(\xi)=2\xi$

$$\therefore\ \psi_1(x)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2}}2\sqrt{\frac{m\omega}{\hbar}}\,xe^{-\frac{m\omega}{2\hbar}x^2}$$

Comparing with $\psi(x)=xe^{-\frac{m\omega x^2}{2\hbar}}$, we have $n=1$

$$\therefore\ E_n=\left(n+\frac{1}{2}\right)\hbar\omega=\frac{3}{2}\hbar\omega$$

**Correct option is (b)**

**Example 16.** The ground state energy of a one-dimensional harmonic oscillator is 6.2 eV. If the oscillator undergoes a transition from its $n = 3$ level to $n = 2$ level by emitting a photon, what is the wavelength associated with the photon ? **[HCU 2011]**

(a) 50 nm (b) 62 nm (c) 100 nm (d) 200 nm

**Soln.** The energy of one-dimensional harmonic oscillator is $E_n = \left(n+\frac{1}{2}\right)h\nu$.

For ground state, $n = 0$. $\therefore$ $E_0 = \frac{h\nu}{2} = 6.2\text{ eV}$ (given).

The energy of the photon is $\frac{hc}{\lambda} = E_3 - E_2 = \frac{7}{2}h\nu - \frac{5}{2}h\nu = h\nu$.

$\Rightarrow \frac{hc}{\lambda} = 12.4\text{ eV} \Rightarrow \lambda = \frac{6.6\times10^{-34}\times3\times10^{8}}{12.4\times1.6\times10^{-19}} \simeq 100\text{ nm}.$

**Correct option is (c)**

**Example 17.** In a harmonic oscillator, if the transition energy from 3rd state to 2nd state is 4.8 eV, what is the ground state energy? **[HCU 2014]**

(a) 1.2 eV (b) 2.4 eV (c) 4.8 eV (d) 0.6 eV

**Soln.** The energy of the $n$th state of a harmonic oscillator is given by $E_n = \left(n+\frac{1}{2}\right)\hbar\omega$.

$\therefore$ $E_0 = \frac{\hbar\omega}{2}$ (ground state); $E_1 = \frac{3\hbar\omega}{2}$ (2nd state); $E_2 = \frac{5\hbar\omega}{2}$ (3rd state)

Given : $E_3 - E_2 = 4.8\text{ eV} \Rightarrow \frac{5}{2}\hbar\omega - \frac{3\hbar\omega}{2} = 4.8\text{ eV} \Rightarrow \hbar\omega = 4.8\text{ eV}$

So the ground state energy $= \frac{1}{2}\hbar\omega = 2.4\text{ eV}$.

**Correct option is (b)**

**Example 18.** A particle of mass $m$ is in a one-dimensional potential given by $V(x) = \infty$ for $x < 0$ and $V(x) = kx^2/2$ for $x > 0$. The first excited state has an energy **[DU 2015]**

(a) $(3/2)\hbar\sqrt{(k/m)}$ (b) $(7/2)\hbar\sqrt{(k/m)}$ (c) $(1/2)\hbar\sqrt{(k/m)}$ (d) $(5/2)\hbar\sqrt{(k/m)}$

**Soln.** Given : $V(x) = \begin{cases} \infty & ; \ x<0 \\ \frac{kx^2}{2} & ; \ x>0 \end{cases}$

The energy for this one sided one-dimensional linear harmonic oscillator is $E = \left(n+\frac{1}{2}\right)\hbar\sqrt{\frac{k}{m}}$, where n = 1, 3, 5, .....

The first excited state corresponds to $n = 3$ whose energy is given by $E_3 = \frac{7}{2}\hbar\sqrt{\frac{k}{m}}$.

**Correct option is (b)**

# Chapter 8

# Nuclear Physics

## 8.1: Nucleons:

All nuclei are composed of two types of particles, positively charged protons and neutral neutrons, referred collectively as nucleons. Their principal properties are listed in Table 7-1.

| | Proton | Neutron |
|---|---|---|
| Charge | $+1.6 \times 10^{-19}$ C | 0C |
| Rest Mass | $1.67252 \times 10^{-27}$ kg<br>938.256 MeV<br>1.007277 u | $1.67482 \times 10^{-27}$ kg<br>939.550 MeV<br>1.008 665 u |
| Spin | 1/2 | 1/2 |
| Magnetic Moment | $+2.7928\,\beta_n$ | $-1.9128\,\beta_n$ |

Atomic nuclei of the same element have the same number of protons but can have different number of neutrons. The electron structure of the atom was understood before even the composition of its nuclear was known. The reason is that the forces that hold the nucleus together are vastly stronger than the electric forces that hold the electrons to its nucleus, and it is correspondingly harder to break apart a nucleus to find out what is inside. Changes in the electron structure of an atom such as those that occur when a photon is emitted or absorbed or when a chemical bond is formed or broken, involve energies of only few electronvolts. Change in nuclear structure. On the other hand, involve energies in MeV, a million times greater.

All the nuclei contain neutron and proton are jointly called nucleon.

The atomic number of an element is the number of protons in each of its atomic nuclei, which is same as the number of electrons in neutral atom of an element. The varieties of element that differ is number of neutron but same number of protons is the nuclei are called its isotopes.

## 8.2. Basic characteristics of the atomic nucleus

**Atomic number, proton number** (Z): Number of protons in the atomic nucleus, hence number of electrons in the neutral atom.

**Neutron number** (N): Number of neutrons in the atomic nucleus.

**Mass number** (A): Total number of nucleons in the nucleus $A = Z + N$.

**Notation:** The atomic number Z is given as left subscript of the atomic symbol X, the mass number A as left superscript, the neutron number N as right subscript: ${}^{A}_{Z}X_{N}$

**even-even nuclei:** even proton number Z, even neutron number N,

**even-odd nuclei:** even proton number Z, odd neutron number N,

**odd-even nuclei:** odd proton number Z, even neutron number N,

**odd-odd nuclei:** odd proton number Z, odd neutron number N.

**Isotopes:** Atomic nuclei with the same atomic number Z, but different neutron numbers N.

- ${}^{A}_{Z}X_{N}$ and ${}^{A+1}_{Z}X_{N+1}$ are isotopes. Example, the carbon isotopes ${}^{12}C$, ${}^{13}C$ and ${}^{14}C$.

Basically, isotopes are chemically equivalent. Only processes that depend on mass exhibit a slightly different behavior for different isotopes (differences in the physical chemical equilibria, differences in diffusion velocity, isotopic shifts in atomic spectra, resonance frequencies in molecules, critical temperature of superconductors). These phenomena are called isotope effects.

**Isobars:** Atomic nuclei with equal mass number A, but different proton numbers Z.

- ${}^{A}_{Z}X_{N}$ and ${}^{A}_{Z+1}Y_{N-1}$ are isobars. Example: ${}^{14}C$ and ${}^{14}N$

**Isotones:** Atomic nuclei with equal neutron number N, but different atomic numbers Z.

- ${}^{A}_{Z}X_{N}$ and ${}^{A+1}_{Z+1}Y_{N}$ are isotones

**Radius of nucleus:**

Shape of atomic nuclei is mostly deformed in an axially symmetric manner, spherical near closed nucleon shells. Nuclear radius R, may be estimated by the formula

$$R = r_0 A^{1/3},\ \text{where } r_0 \approx 1.2\ fm = 1.2\times10^{-15}\ m,\ A = \text{atomic mass number}$$

**Binding Energy:**

Binding energy, B, the energy released when free nucleons are bound together to form a nucleus. SI unit is the joule (J). Usually the binding energy is given in MeV:

$$1\ MeV = 1.6022\times10^{-13}\ J$$

The mass of a stable atomic nucleus is smaller than the sum of the masses of the constituent nucleons.

Mass defect, $\Delta W(A,Z)$ the difference between the sum of masses of all nucleons and the nuclear mass $m_K(A,Z)$:

$$\Delta W(A,Z) = m_p Z + (A-Z)m_n - m_K(A,Z)$$

According to mass-energy equivalence, the mass defect ΔW is related to the binding energy B,

$$B = \Delta W(A,Z)c^2,\qquad 1\,\text{MeV}/c^2 = 1.7827\times10^{-30}\ \text{kg}$$

**Atomic Mass Unit (u):**

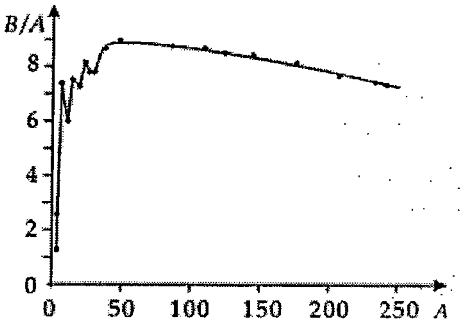
It is equal to 1/12 of the mass of a neutral atom of the carbon isotope ${}^{12}$C:

$$u = \frac{1}{12} m\ {}^{12}C = \frac{1\,g}{N_A} = 1.660\times10^{-27}\ kg \quad (N_A\text{: Avogadro's number})$$

| Quantity | Symbol | Value | Error /ppm |
|---|---|---|---|
| atomic mass unit | $u$ | 931.494 32 MeV/$c^2$ | 0.30 |
| electron mass | $m_e$ | 0.510 999 06 MeV/$c^2$ | 0.30 |
| muon mass | $m_\mu$ | 105.658 389 MeV/$c^2$ | 0.32 |
| proton mass | $m_p$ | 938.272 31 MeV/$c^2$ | 0.30 |
| neutron mass | $m_n$ | 939.565 63 MeV/$c^2$ | 0.30 |
| Planck's constant | $\hbar$ | $6.582\,122\,0\cdot10^{-22}$ MeV · s | 0.30 |

**Binding energy per nucleon:**

Binding energy per nucleon is not constant. It increases initially, become highest for Fe, then decreases.



**Binding energy per nucleon** ( B/A): A measure for the stability of an atomic nucleus. Mean experimental value: $B/A \approx 8\ \text{MeV}$.

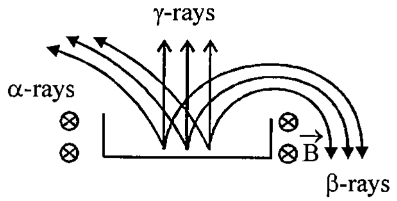
For light nuclei, the binding energy per nucleon increases with the mass number. The most stable atomic nucleus is iron ($^{56}$Fe) with a binding energy per nucleon of $\approx 8.8$ MeV . For A $> 56$, the binding energy per nucleon decreases with increasing nucleon number.

Therefore, nuclear energy may be released either by fusion of light nuclei or by fission of heavy nuclei.

The local maxima of the binding energy in the range of light nuclei $\left(\text{e.g. for } {}^4_2He\right)$ are caused by closure of neutron and/or proton shells, analogous to the strong binding of the electron shell in inert gas atoms.
Saturation of nuclear forces, the binding energy per nucleon is approximately constant at about 8 MeV.
The magnitude of the binding energy of a nucleus determines its stability against decay.

## 8.3. Radioactivity:

Radioactivity is the phenomenon of spontaneous distintegration, attended with emission of corpuscular or electromagnetic radiations, of heavy atomic nuclei $\left(\text{for } \dfrac{N}{Z} > 1.5\right)$ like uranium, radium, etc., at a constant rate unaffected by any physical or chemical changes or influences such as temperature, pressure etc. to which the atom may be subject. It is a nuclear property of the active element and in all radioactive processes a transmutation of the element occure an altogether new nucleus is formed.



**Radioactive Decay law:**

**1.** On emission of $\alpha$ or $\beta$ rays, which is usually but not invariably accompanied by $\gamma$-ray emission, the emitting parent nuclide transforms into a new daughter element. The daughter element again is radioactive so that the process of successive disintegration continuous till the original active parent nucleid gets transformed into stable one.

**2.** The rate of radioactive disintegration, that is, the number of atoms that break up at any instant of time 't' is directly proportional to the number $N_t$ of active nuclides present in the sample under study at that instant.

**Decay equation:**

Let $N_t$ be the number of active nuclides present in the sample at any time '$t$' then we have experimentally,

$$-\frac{dN_t}{dt} \propto N_t$$

$$\Rightarrow \quad \frac{dN_t}{dt} = -\lambda N_t$$

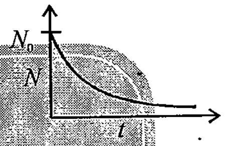
Where, $\lambda$, the constant of proportionality, is known as the *decay* constant – a characteristic constant of the element (nuclide). The negative sign hints at the fact that $N_t$ decreases with $t$.

$$\frac{dN_t}{N_t} = -\lambda dt$$

$$\ell n\, N_t = -\lambda t + A$$

where, $A$ is the constant of integration.

At $t = 0$, $N_t = N_0$, the initial number of nuclides and $A = \ell n\, N_0$



Therefore, $$\ell n\left(\frac{N_t}{N_0}\right) = -\lambda t$$

Or, $$\boxed{N_t = N_0 e^{-\lambda t}}$$

**Half-life:** The half-life of a radioactive nuclide is defined as the time $T_{½}$ in which the original amount of radioactive atoms is reduced by way of distintegration to half its value.

At $t = T_{½}$, $N_t = N_0/2$

$$\frac{N_0}{2} = N_0 e^{-\lambda T_{½}}$$

$$\therefore \quad T_{½} = \frac{\ell n 2}{\lambda} = \frac{2.303 \log 2}{\lambda} = \frac{0.693}{\lambda}$$

Or, $$\boxed{\lambda T_{½} = 0.693 = \text{constant}}$$

**Average life:**

The average or mean life time $\overline{T}$ of a radioelement is the average life time of all the atoms in the given sample and is defined as the ratio of the total life time of all the atoms to be total number of atoms.

$$\therefore \quad \overline{T} = \frac{t_1 dN_1 + t_2 dN_2 + \ldots\ldots}{dN_1 + dN_2 + \ldots\ldots} = \frac{\Sigma t dN}{\Sigma dN} = \frac{\int_{t=0}^{\infty} t dN}{\int_{N_0}^{0} dN} = \frac{\int_{t=0}^{\infty} t dN}{-N_0}$$

But, we have,

$$dN = d\left(N_0 e^{-\lambda t}\right) = -\lambda N_0 e^{-\lambda t} dt$$

$$\therefore \quad \overline{T} = \lambda \int_0^{\infty} t e^{-\lambda t} dt = \lambda \left[ -\frac{t}{\lambda} e^{-\lambda t} + \frac{1}{\lambda} \int_0^{\infty} e^{-\lambda t} dt \right]$$

$$= \lambda\left[-\frac{t}{\lambda}e^{-\lambda t} - \frac{1}{\lambda^2}e^{-\lambda t}\right]_0^{\infty} = \frac{1}{\lambda}$$

Therefore, $\boxed{\bar{T} = \frac{1}{\lambda}}$

We can write also, $\boxed{T_{1/2} = \frac{0.693}{\lambda} = 0.693\bar{T}}$

**Activity:**

The activity or strength $A_t$ of a radioactive sample at any instant $t$ is thus defined as the number of disintegration occurring in the sample in unit time at $t$, that is,

Activity, $\boxed{A_t = \left|\frac{dN_t}{dt}\right| = \lambda N_t = \frac{0.693}{T}N_t}$

**Units of activity:**

The customary unit of radioactivity is called the curie (Ci). It is defined as the acitivity of any radioactive substance that disintegrates at the rate of $3.7 \times 10^{10}$ disintegration per second.

$$1\ \text{Ci} = 1\ \text{curie} = 3.7 \times 10^{10}\ \text{disintegration/sec}$$

$$1\ \text{mCi} = 10^{-3}\ \text{curie}$$

$$1\ \mu\text{Ci} = 10^{-6}\ \text{curie}$$

**Radioactive displacement law:**

**1.** The $\alpha$ particles having been identities as helium nuclei. Suppose I have parent nucleus having mass number A and atomic number, after $\alpha$ decay the parent nucleus become a new daughter element whose atomic number $Z-2$ and mass $A-4$. It position in periodic table will be shifted towards left by two.

$${}^{A}X_{Z} \xrightarrow{\alpha} {}^{A-4}Y_{Z-2}$$

**2.** The $\beta$-particle defined as ${}_{-1}e^{0}$ due to $\beta$ emission the parent elements $(Z-A)$ become a daughter element $(Z+1, A)$. So, it position in periodic table will be shifted towards right by one.

$${}^{A}X_{Z} \xrightarrow{\beta} {}^{A}Y_{Z+1}$$

**Natural radioactive decay series:**

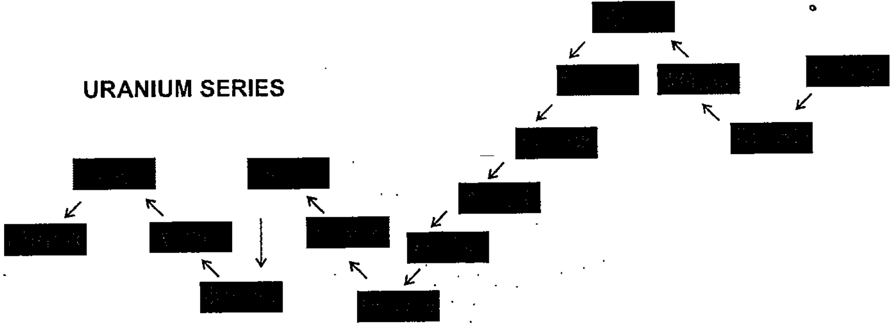
There are three naturally occuring radioactive series. These three series are

(i) Uranium series: $(4n+2)$

(ii) Actinium series: $(4n+3)$

(iii) Thorium series: $(4n)$

URANIUM SERIES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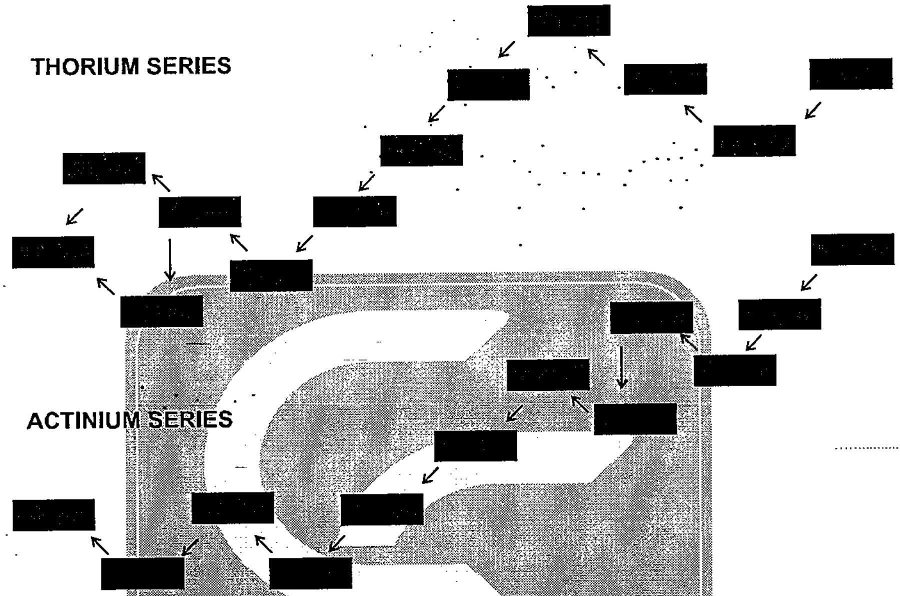
THORIUM SERIES

ACTINIUM SERIES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**Note:** $\nwarrow -(\beta\text{-decay})$, $\swarrow -(\alpha\text{-decay})$

**Successive transformation:**

Consider a radioactive nuclide *P, symbolised by script 1, to decay into another radioactive nuclide Q (script 2); the latter again decays into a stable end-product R (script 3). For instance,

$$P(1)\xrightarrow{\lambda_1}Q(2)\xrightarrow{\lambda_2}R(3)\ [stable]$$

Let $\lambda_1, \lambda_2$ be the decay constants of nuclides 1 and 2 respectively, and $N_1, N_2, N_3$ be the number of atoms of the three kinds at any intant $t$.

Now the second nuclide would be formed at the rate $\lambda_1 N_1$ by the decay of the parent atom and disappear by its own decay at the rate $\lambda_2 N_2$; the atoms of the end-product i.e. R appear at the rate $\lambda_2 N_2$ by the decay of nuclide 2, but being stable do not disappear. So,

$$\frac{dN_1}{dt} = -\lambda_1 N_1 \qquad \text{... (i)}$$

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2 \qquad \text{... (ii)}$$

$$\frac{dN_3}{dt} = \lambda_2 N_2 \qquad \text{... (iii)}$$

Solving equation (i) : $(N_1)_t = N_{10}e^{-\lambda_1 t}$

where $N_{10}$ is the number of atoms of nuclide 1, at time t = 0

From equation (ii), using (iii), we obtain

$$\frac{dN_2}{dt} = \lambda_1 N_{10}\, e^{-\lambda_1 t} - \lambda_2 N_2$$

Or, $$\frac{dN_2}{dt} + \lambda_2 N_2 = \lambda_1 N_{10} e^{-\lambda_1 t} \qquad \text{... (iv)}$$

Multiplying both sides of equation (iv) by $e^{\lambda_2 t}$, the integrating factor,

$$e^{\lambda_2 t}\left(\frac{dN_2}{dt} + \lambda_2 N_2\right) = \lambda_1 N_{10}\, e^{(\lambda_2-\lambda_1)t}$$

Or, $$\frac{d}{dt}\left(N_2 e^{\lambda_2 t}\right) = \lambda_1 N_{10} e^{(\lambda_2-\lambda_1)t}$$

Integrating,

$$N_2 e^{\lambda_2 t} = \frac{\lambda_1}{\lambda_2-\lambda_1} N_{10} e^{(\lambda_2-\lambda_1)t} + A \qquad \text{... (v)}$$

Where A is the constant of integration,

Now at $t = 0, (N_2)_t = N_{20} =$ constant. So, from equation (v),

$$A = N_{20} - \frac{\lambda_1}{\lambda_2-\lambda_1} N_{10}$$

$$\therefore \quad (N_2)_t = \frac{\lambda_1}{\lambda_2-\lambda_1} N_{10}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right) + N_{20} e^{-\lambda_2 t} \qquad \text{... (vi)}$$

From equation (iii), using (iv), we obtain on integration,

$$(N_3)_t = \left(\frac{\lambda_1}{\lambda_2-\lambda_1} N_{10} - N_{20}\right) e^{-\lambda_2 t} - \frac{\lambda_2}{\lambda_2-\lambda_1} N_{10} e^{-\lambda_1 t} + B \qquad \text{... (vii)}$$

At $t = 0, (N_3)_t = N_{30}$. So, from equation (vii), we get

$$\because \quad B = N_{30} + N_{20} + N_{10}$$

$$\therefore \quad (N_3)_t = N_{30} + N_{20}\left(1 - e^{-\lambda_2 t}\right) + N_{10}\left(1 + \frac{\lambda_1}{\lambda_2-\lambda_1} e^{-\lambda_2 t} - \frac{\lambda_2}{\lambda_2-\lambda_1} e^{-\lambda_1 t}\right) \qquad \text{... (viii)}$$

Equations (i), (vi) and (viii) therefore, constitute the solution of the problem.

In most cases, only the first nuclide is present at $t = 0$. So, $N_{20} = N_{30} = 0$. We thus have from (vi) and (viii).

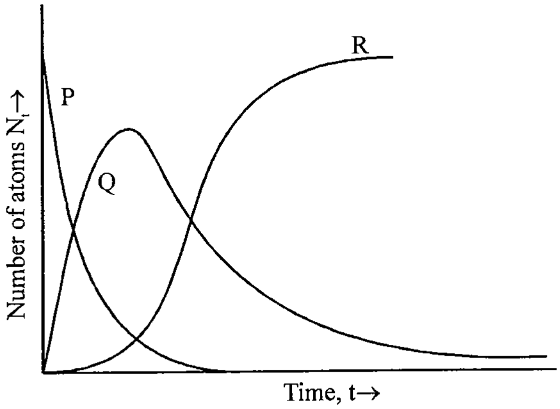
$$(N_2)_t = \frac{\lambda_1}{\lambda_2-\lambda_1} N_{10}\left(e^{-\lambda_1 t} - e^{-\lambda_2 t}\right)$$

$$(N_3)_t = N_{10}\left(1 + \frac{\lambda_1}{\lambda_2-\lambda_1} e^{-\lambda_2 t} - \frac{\lambda_2}{\lambda_2-\lambda_1} e^{-\lambda_1 t}\right)$$

The decay of the first nuclide P, the growth and decay of the second nuclide Q and the growth of the third nuclide R are shown in the figure below .

**Time for Q(2) to attain a maximum:**

Equation (vi) gives how $N_2$ varies with $t$. It shows that $N_2 = 0$ at $t = 0$. It increases with increasing $t$ and attains a maximum at $t = t_m$, say.



Now, we have, $\dfrac{dN_2}{dt} = \dfrac{\lambda_1 N_{10}}{\lambda_2 - \lambda_1}\left(-\lambda_1 e^{-\lambda_1 t} + \lambda_2 e^{-\lambda_2 t}\right)$

$$\therefore \quad \frac{dN_2}{dt} = 0 \Rightarrow \lambda_2 e^{-\lambda_2 t_m} = \lambda_1 e^{-\lambda_1 t_m}$$

$$\therefore \quad e^{(\lambda_1 - \lambda_2)t_m} = \frac{\lambda_1}{\lambda_2}$$

Or, $(\lambda_1 - \lambda_2)t_m = \ell n \dfrac{\lambda_1}{\lambda_2}$

$$\therefore \quad t_m = \frac{1}{\lambda_1 - \lambda_2}\ell n \frac{\lambda_1}{\lambda_2}$$

That is, $t_m$ corresponds to maximum $N_2$ can be readily verified by obtaining the second time derivative of $N_2$ which becomes negative at $t = t_m$.

**Branching of radioactivity :**

When a radio-element shows that $\alpha$ and $\beta$ activities, there are definite probabilities of $\alpha$ and $\beta$ decay constants for the branches. Let it contain $N$ atoms at any time $t$, the total rate of disintegration will be

$$\frac{dN}{dt} = -\left(\lambda_\alpha + \lambda_\beta\right)N = \lambda N, \text{ where } \lambda = \lambda_\alpha + \lambda_\beta \text{ the total decay constant.}$$

If $(T_{1/2})_\alpha, (T_{1/2})_\beta$ be the partial half lives of $\alpha$ and $\beta$ decay.

$$\lambda_\infty = \frac{0.693}{(T_{1/2})_\alpha}, \quad \lambda_\beta = \frac{0.693}{(T_{1/2})_\beta}$$

$$\therefore \quad \lambda = 0.693\left(\frac{1}{(T_{1/2})_\alpha} + \frac{1}{(T_{1/2})_\beta}\right)$$

But if $T_{1/2}$ be the mean half-life of the substance, then

$$\lambda = \frac{0.693}{T_{1/2}}$$

$$\therefore \quad \boxed{\frac{1}{T_{1/2}} = \frac{1}{(T_{1/2})_\alpha} + \frac{1}{(T_{1/2})_\beta}}$$

**Determination of constants of radioactivity:**

**1. For simple decays :** We have seen that the activity $A$ of a sample exponentially with time according to the equation,

$$A_t = A_0\, e^{-\lambda t}.$$

$$\therefore \quad \ell n\, A_t = \ell n A_0 - \lambda t$$

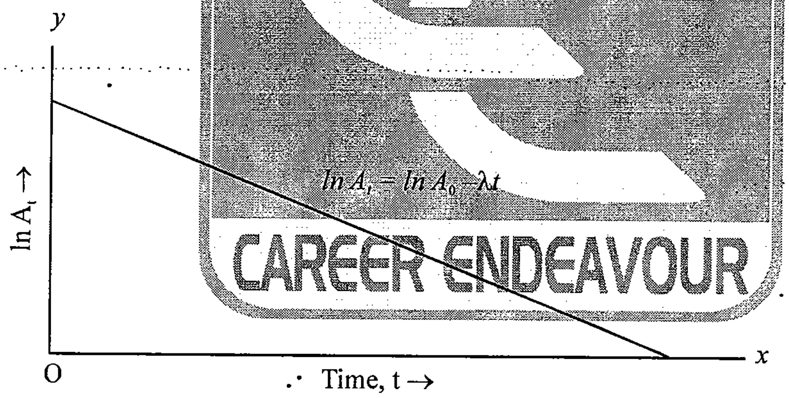
Or, $\quad y = -\lambda t + C \qquad \ldots \text{(i)}$

where, $y = \ell n\, A_t$ and $C = \ell n\, A_0$.

Differentiating (i),

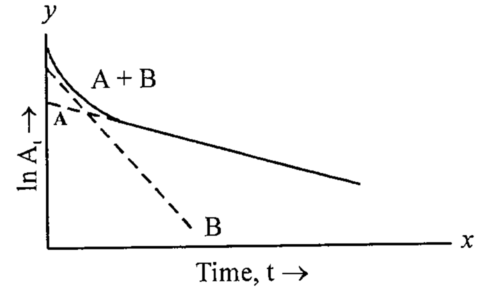
$$\frac{dy}{dt} = -\lambda$$

So, when the logarithm of the activity, $\ln A_t$, is plotted against time $t$, it will give a straight line with a slope equal to $-\lambda$. This therefore would furnish us a method for evaluating $\lambda$, provided we can measure the activity of the sample.



**2. Complex decays:**

We conclude the section by indicating how one can determine half-lives for complex decays, e.g. two half-lives when a sample contains a mixture of say two activities $A$ and $B$ such that $T_{1/2}(A) >> T_{1/2}(B)$. A graph of the logarithm of the observed activity of the sample is first plotted against time, the observations however are extended over a time large enough compared to $T_{1/2}(B)$. The tail-end of the curve is due to the activity $A$, since the contribution from $B$ in this region is negligible. So, if the straight part of the graph is extrapolated to $t = 0$, the contribution to the total observed activity of the sample from A at any instant $t$ can be found. Subtracting this activity of A from the observed total activity at $t$, we get the activity of $B$ at $t$. Once the contributions from the two activities are known, their half-lives can be evaluated by the usual procedure.



- $\alpha$**-decay:**

- **Velocity of** $\alpha$**-particles:**

  Let '$v$' be the velocity of the $\alpha$-particles, $M$ be its mass, $q$ the charge, $r$ the radius of the track and $B$ the magnetic field, then $v = Bqr/M$.

- **Range of** $\alpha$**-particles:**

  The distance through which an $\alpha$-particle travels in a specified material before stopping to ionise it is called the range of $\alpha$-particles in that material. The range is thus a sharply defined ionisation path-length.

  The range depends on (i) The initial energy of the $\alpha$-particles, (ii) the ionisation potential of the gas and (iii) the chances of collision between the $\alpha$-particles and on the nature and the temperature and pressure of the gas. With increase of pressure, the range decreases; it increases if the temperature of the gas increased.

  The range R in standard air is proportional to $v^3$, i.e. $R \propto v^3 \Rightarrow R = av^3$

  The relation is known as the Geiger law, which is valid only in a limited velocity range.

  Since $R \propto v^3$ and the energy $E = \frac{1}{2}mv^2$, the range-energy relationship is

  $$R \propto E^{3/2} \Rightarrow R = bE^{3/2}$$

- **Specific ionisation:**

  The number of ion-pairs formed per unit path length at any point in the path of the $\alpha$-particle is called specific ionisation and is symbolised by I.

  Since, $E \propto R^{2/3} \Rightarrow \frac{dE}{dR} \propto R^{-\frac{1}{3}} \propto \frac{1}{v}$

- **Geiger-Nuttall law:**

  An important quantitative relation between the range $R$ of the $\alpha$-particles and the decay constant $\lambda$ of the emitting nuclei was experimentally discovered by Geiger and Nuttall(1911) and is called the Geiger-Nutall law. The relation runs as:

  $$\ln \lambda = A + B \ln R,$$

  where $A$ and $B$ are constants having values different for different radioactive series.

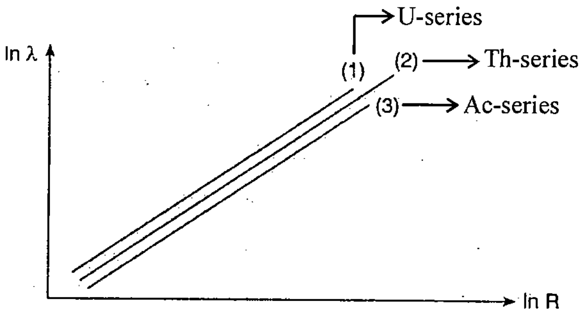


**Figure:** Variation of ln λ with *l*nR: Geiger-Nuttall law

Since $R \propto E^{3/2} \Rightarrow \ln \lambda = C + D \ln E$,
where $C$, $D$ are two constants.

- **$\alpha$-disintegration energy:**

$$ {}_{Z}^{A}X \to {}_{Z-2}^{A-4}Y + {}_{2}^{4}He $$

The $Q$-value of the decay process is known as the $\alpha$-disintegration energy which is the total energy released in the disintegration process and is given by $Q_\alpha = (M_X - M_\alpha - M_Y)c^2$
where $M$'s are the masses of the particles and '$c$' the velocity of light in vacuum.

For heavy nuclei, $Q_\alpha$ is positive, so the decay can occur spontaneously as it does. According to laws of conservation of momentum and energy.

$$0 = M_\alpha v_\alpha - M_Y v_Y$$

and $Q_\alpha = \frac{1}{2} M_\alpha v_\alpha^2 + \frac{1}{2} M_Y v_Y^2$

$$\Rightarrow Q_\alpha = \frac{1}{2} M_\alpha v_\alpha^2 \left(1 + \frac{M_\alpha}{M_Y}\right) = T_\alpha \left(1 + \frac{M_\alpha}{M_Y}\right)$$

Kinetic energy of ejected $\alpha$-particles, $T_\alpha = \frac{Q_\alpha}{1 + M_\alpha / M_Y}$

$$\Rightarrow Q_\alpha = T_\alpha \frac{[My + M_\alpha]}{M_y} = T_\alpha \left[\frac{A}{A-4}\right]$$

$$\Rightarrow \boxed{T_\alpha = Q_\alpha \left[1 - \frac{4}{A}\right]} \;\&\; T_d = Q_\alpha - T_\alpha = \frac{4Q}{A}$$

Where K.E. of $\alpha$-particle $= T_\alpha$
As $A \sim 200 \Rightarrow T_\alpha \sim Q_\alpha$

**$\alpha$-decay through tunnelling:**

$$ {}_{Z}X^{A} \longrightarrow {}_{Z-2}Y^{A-4} + 2\text{He}^4 $$

$$R = R_0 A^{1/3}$$

$$V(R) = \frac{2(Z-2)|e|^2}{4\pi \in_0 R}$$

$$T = e^{-G}, G = A(Z-2)E^{-1/2} - B(Z-2)^{1/2} R^{1/2}$$

$$A = 1.587\times10^{-6}\ \text{J}^{1/2}$$

$$B = 94\times10^{6}\ m^{-1/2}$$

$$v = \frac{v}{2R};\ \lambda = vT;\ T_{½} = \frac{\ln 2}{\lambda}$$

**Example:** ${}_{92}U^{238} \longrightarrow {}_{90}Th^{234} + {}_{2}He^{4}$ and the energy of the emitted $\alpha$ particle is $E = 6.72\times10^{-13}$ joule. Determine the half-life time of ${}_{92}\text{U}^{238}$.

**Soln.** Nuclear radius of parent

$$R = R_0 A^{1/3} = (1.4\,\text{fm})(238)^{1/3} = 8.7\,\text{fm} = 8.7\times10^{-15}\,\text{m}$$

Atomic number of parent $Z = 92$

Tunnelling probability,

$$T = e^{-G} = e^{-\left(1.587\times10^{-6}\,\text{J}^{1/2}\right)(92-2)\left(6.72\times10^{-13}\,\text{J}\right)^{-1/2} + \left(94\times10^{6}\,\text{m}^{-1/2}\right)(92-2)^{1/2}\left(8.7\times10^{-15}\,\text{m}\right)^{1/2}} = \frac{6}{10^{39}}$$

Out of $10^{39}$ strikes $\alpha$ emerges out six times only.

Velocity, $v = \sqrt{\frac{2E}{m}} = \sqrt{\frac{2\left(6.72\times10^{-13}\,\text{J}\right)}{4\times1.67\times10^{-27}\,\text{kg}}} = 1.42\times10^{7}\ \text{ms}^{-1}$

Collision frequency, $v = \frac{v}{2R} = \frac{1.42\times10^{7}\ \text{ms}^{-1}}{2\left(8.7\times10^{-15}\,\text{m}\right)} = 8.2\times10^{20}\,\text{s}^{-1}$

Decay constant, $\lambda = vT = \left(8.2\times10^{20}\,\text{s}^{-1}\right)\left(6\times10^{-39}\right) = 4.92\times10^{-18}\,\text{s}^{-1}$

Half life $T_{½} = \frac{\ln 2}{\lambda} = \frac{\ln 2}{4.92\times10^{-18}\,\text{s}^{-1}} = 1.4\times10^{17}\,\text{s}$

$$= \frac{1.4\times10^{17}}{365\times24\times60\times60}\ \text{years} = 4.4\times10^{9}\ \text{years}$$

The experimental value of half life is $4.5\times10^{9}$ years. So, we have obtained a rough estimate of half life of ${}_{92}U^{238}$.

- **β–decay:**

Let $v$ be the velocity of a given β-particle, $B$ the magnetic flux density, $m$ the relativistic mass of the β-particle and $r$ the radius of the circular track.

Then $\frac{mv^2}{r} = Bev \Rightarrow r = \frac{mv}{Be} = \frac{m_0}{\sqrt{1-v^2/c^2}}\left(\frac{v}{Be}\right)$

- **β - decay , we write:**

$${}^{A}_{Z}X \to {}^{A}_{Z+1}Y + {}^{0}_{-1}e$$

The disintegration energy in $\beta^-$decay is:

$$Q_{\beta^-} = \left[M_n(A,Z) - M_n(A,Z+1) - m_e\right]c^2$$

$$= \left[M(A,Z) - Zm_e - M(A,Z+1) + (Z+1)m_e - m_e\right]c^2 \text{ (in terms of atomic mass)}$$

$$= \left[M(A,Z) - M(A,Z+1)\right]c^2$$

where $M_n$ is the nuclear mass, $M$ the atomic mass and $m_e$ the mass of electron.

$Q_{\beta^-} > 0$, if $M(A, Z) > M(A, Z + 1)$

implying that $\beta^-$ decay occurs only if the mass of the parent atom is greater than that of the daughter atom.

- **$\beta^+$ - decay:**

$${}^{A}_{Z}X \to {}^{A}_{Z-1}Y + {}^{0}_{+1}e$$

$$\Rightarrow \quad Q_{\beta^+} = \left[M_n(A,Z) - M_n(A,Z-1) - m_e\right]c^2$$

$$= \left[M(A,Z) - Zm_e - M(A,Z-1) + (Z-1)m_e - m_e\right]c^2$$

$$= \left[M(A,Z) - M(A,Z-1) - 2m_e\right]c^2$$

For $\beta^+$ - decay to occur: $Q_{\beta^+} > 0$, if $M(A,Z) > M(A,Z-1) + 2m_e$

i.e. the mass of the parent atom is greater than the daughter atom at least twice the electronic mass, i.e. 1.02 MeV.

- **Electron capture:**

$${}^{A}_{Z}Y + {}^{0}_{-1}e \to {}^{A}_{Z-1}Y$$

Therefore, disintegration energy, $Q_e = \left[M_n(A,Z) + m_e - M_n(A,Z-1)\right]c^2 - B_e$ where $B_e$ is the binding energy of the electron to the orbit.

$$\Rightarrow \quad Q_e = \left[M(A,Z) - Zm_e + m_e - M(A,Z-1) + (Z-1)m_e\right]c^2 - B_e$$

$$= \left[M(A,Z) - M(A,Z-1)\right]c^2 - B_e$$

For electron capture to occur: $Q_e > 0$, if $M(A,Z) > M(A,Z-1) + B_e$

i.e. the mass of the parent atom is greater than that of the daughter atom by at least the binding energy of the electron.

- **Selection rule for $\beta^+$ - decay:**

  If $\ell_\beta$ is odd, initial and final nuclei must have opposite parities (parity changes in these transitions); for even $\ell_\beta$ values the initial and final nuclei must have same parity (no change in parity). Furthermore, as in allowed transitions, the emission of leptons (electron and neutrino) in the singlet state (Fermi-selection rule) requires $\Delta I \le \ell_\beta$, whereas triplet-state (G-T selection rule) emission requires $\Delta I \le \ell_\beta + 1$. Thus selection rules for forbidden transitions are:

  **First forbidden** – For these transitions $\ell_\beta = 1$ and parity changes.

  Fermi-selection rules: $\Delta I = \pm 1, 0\,(\text{except } 0 \to 0)$

  Gamow Teller rules: $\Delta I = \pm 2, \pm 1, 0\left(\text{except } 0 \to 0, \frac{1}{2} \to \frac{1}{2}, 0 \leftrightarrow 1\right)$

  The examples are:

| | | |
|---|---|---|
| ${}^{39}_{18}Ar \to {}^{39}_{19}K + \beta^-$ | $(7/2^- \to 3/2^+)$ | *GT.* |
| ${}^{14}_{6}C \to {}^{14}_{7}N + \beta^-$ | $0 \to 1$ | *F* |
| ${}^{38}_{17}Cl \to {}^{38}_{18}Ar + \beta^-$ | $(2^- \to 2^+)$ | *F + GT mixed* |
| ${}^{139}_{56}Ba \to {}^{139}_{57}La + \beta^-$ | $(7/2^- \to 7/2^+)$ | *F + GT mixed* |
| ${}^{143}_{59}Pr \to {}^{143}_{60}Nd + \beta^-$ | $(7/2^+ \to 7/2^-)$ | *F + GT mixed* |
| ${}^{147}_{61}Pm \to {}^{147}_{62}Sm + \beta^-$ | $(7/2^+ \to 7/2^-)$ | *F + GT mixed* |
| ${}^{85}_{36}Kr \to {}^{85}_{37}Rb + \beta^-$ | $(9/2 \to 5/2)$ | *GT.* |
| ${}^{89}_{38}Sr \to {}^{89}_{39}Y + \beta^-$ | $(5/2^+ \to 1/2^-)$ | *GT.* |
| ${}^{137}_{55}Cs \to {}^{136}_{56}Ba^* + \beta^-$ | $(7/2^+ \to 11/2^-)$ | *GT.* |

**Second forbidden.** For these transition $\ell_\beta = 2$ and no change in parity.

Fermi-selection rules: $\Delta I = \pm 2, \pm 1\,(\text{except } 0 \leftrightarrow 0)$

Gamow Teller rules: $\Delta I = \pm 3, \pm 2\,(\text{except } 0 \leftrightarrow 2)$

The examples are:

| | | |
|---|---|---|
| ${}^{137}_{55}Cs \to {}^{137}_{56}Ba + \beta^+$ | $(7/2^+ \to 3/2^+)$ | *F + GT mixed* |
| ${}^{10}_{4}Be \to {}^{10}_{5}B + \beta^-$ | $(0^+ \to 3^+)$ | *GT.* |
| ${}^{22}_{11}Na \to {}^{22}_{10}Ne + \beta^+$ | $(3^+ \to 0^+)$ | *GT.* |

- **γ-decay:**

  In γ-decay there is no change in *Z* and *A*. The γ-ray photons are emitted when the nucleus jumps from an excited state to the lower state (after $10^{-14}$ sec). It is always not possible that when nucleus jumps from higher energy state to lower γ-photon will be emitted.

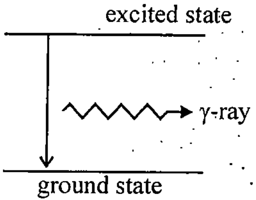
  Sometimes the nucleus in higher energy state directly give energy to the atomic $e^-$ and hence an $e^-$ may come out rather than a γ-ray photon. This process is known as internal conversion and the $e^-$ so emitted is called as conversion $e^-$.

### Energy of emitted $\gamma$-ray in $\alpha$-decay:

A parent nuclei ${}^{A}X_{Z}$, when it emit a particle it goes to the excited state and it comes down to its ground state by emitting $\gamma$-ray.

According energy conservation rule we can write,

$$E_\gamma = Q_\alpha - (T_\alpha + T_y) = Q_\alpha - T_\alpha\left(1+\frac{M_\alpha}{M_y}\right)$$



$$\therefore \quad \boxed{E_\gamma = Q_\alpha - T_\alpha\left(1+\frac{M_\alpha}{M_y}\right)}$$

- **Selection rule for γ-decay:**

| Type | Symbol | Change in Angular momentum, $L$ | Parity change |
|---|---|---|---|
| Electric dipole | $E_1$ | 1 | Yes |
| Magnetic dipole | $M_1$ | 1 | No |
| Electric quadrupole | $E_2$ | 2 | No |
| Magnetic quadrupole | $M_2$ | 2 | Yes |
| Electric octupole | $E_3$ | 3 | Yes |
| Magnetic octupole | $M_3$ | 3 | No |
| Electric $2^L$ – pole | $E_L$ | $L$ | [No for even $L$. Yes for odd $L$] |
| Magnetic $2^L$ – pole | $M_L$ | $L$ | [Yes for even $L$. No for odd $L$] |

### γ-ray interaction with matter :

There are three fundamental procedure in which $\gamma$-ray interact with matter. These three processes

(i) Compton scattering

(ii) Photoelectric effect

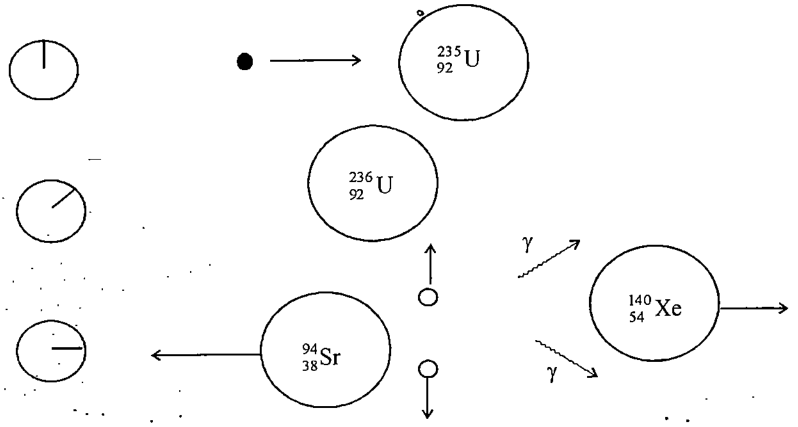
(iii) Pair production

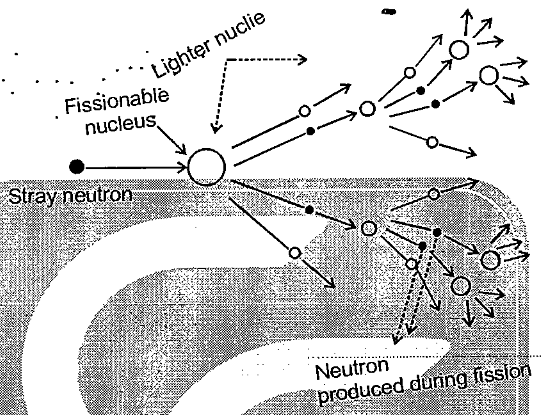
The three factors should therefore be combined into one single (total) absorption coefficient, μ in the absorption law. The intensity of the transmitted $\gamma$-ray is given by

$\boxed{I = I_0 e^{-\mu x}}$ where, $x$ is the thickness of medium.

## 8.4. Nuclear fission:

A nucleus of the uranium isotope ${}^{235}_{92}\text{U}$ undergoes fission when struck by a neutron. It is not the impact of neutron that has this effect. Instead, the ${}^{235}_{92}\text{U}$ nucleus absorbs the neutron to became ${}^{236}_{92}\text{U}$, and the new nucleus is so unstable that almost at once it explodes into two fragments. Later several other heavy nuclides were found to be fissionable by neutrons in similar processes.

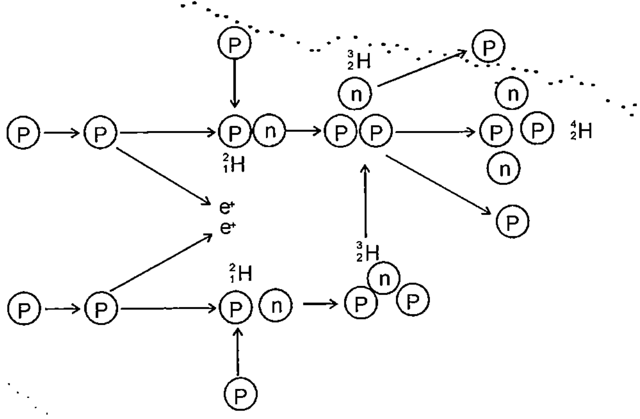

$^{235}_{92}$U
$^{236}_{92}$U
γ
$^{140}_{54}$Xe
$^{94}_{38}$Sr
γ


Lighter nuclie
Fissionable nucleus
Stray neutron
Neutron produced during fission

Reaction is self sustaining if atleast one neutron from each fission event on the average induces another fission event.

## 8.5. Nuclear fission:

Self-sustaining fusion reactions can occur only under condition of extremely high temperature and density. The high temperature ensures that some nuclei-those in the high velocity tail of the Maxwell-Boltzamann distribution have the energy needed to come close enough together to interact, which they do by tunneling through the electric potential barrier between them. (At $10^7$ K temperature, typical of sun is interior, the average proton KE is only about 1KeV, whereas the barrier is about 1MeV). The high density ensures that such collisions are frequent.


P
n
$^2_1$H
$^3_2$H
$^4_2$H
e⁺
e⁺

The combination of four hydrogen nuclie to form a helium nucleus and two positrons.

- **Classification of nuclear reactions:**

(i) **Elastic scattering:** In elastic scattering, the same particles are scattered in different directions and there is no loss of energy. The residual nucleus is the same as the target nucleus and is left in the same state (ground state) as the latter so that it can be represented as X(x, y)X. An example is the scattering of neutrons by graphite: $^{12}\text{C}(n,n)\,^{12}\text{C}$. **Example:** $_2\text{He}^4 + _{79}\text{Au}^{197} \rightarrow _{79}\text{Au}^{197} + _2\text{He}^4$

(ii) **Inelastic scattering:** In inelastic scattering, the same particles are scattered in different directions with different energy, as there is loss of energy due to collision. The residual nucleus which is the same as target nucleus is left in an excited state so that the process can be represented as $X(x,y)X^*$. An example is the collision of fast neutrons with U-238.

(iii) **Radiative capture:** In radiative capture, the incident particle is absorbed or captured by the target nucleus to form the excited compound nucleus which disintegrates to produce one or more $\gamma-$photons and goes down to the ground state. The process may be represented as $X(x,\gamma)Y^*$. **Example:** $_{92}\text{U}^{235} + _0n^1 \longrightarrow _{92}\text{U}^{236}$.

(iv) **Reaction of transformation:** Here the oncoming particle is retained in the nucleus and the compound nucleus emits a different particle so that the product nucleus is different from target nucleus, e.g. $^9\text{Be}(\alpha,n)^{12}\text{C}$

(v) **Photo-disintegration:** In photo-disintegration, a very energetic photon is absorbed by the target nucleus so that it is raised to an excited state and subsequently disintegrates. It can be represented as $X(\gamma,y)Y$. **Example:** $_1\text{H}^2 + \gamma \longrightarrow _1\text{H}^1 + _0n^1$.

(vi) **Stripping reactions:** In stripping reactions, one or more nucleons from the projectile are captured by the target nucleus, the remaining stripped nucleus is emitted in a different direction.

**Example:** $_{29}\text{Cu}^{63} + _1d^2 \longrightarrow _{29}\text{Cu}^{64} + _1\text{H}^1$

(vii) **Direct reactions:** A collision of an incident particle with the nucleus may immediately pull one of the nucleons out of the target nucleus and is called 'pick up reaction'.

**Example:** $_1\text{H}^1 + _6\text{C}^{13} \longrightarrow _1d^2 + _6\text{C}^{12}$

- **Conservation in Nuclear reactions:**

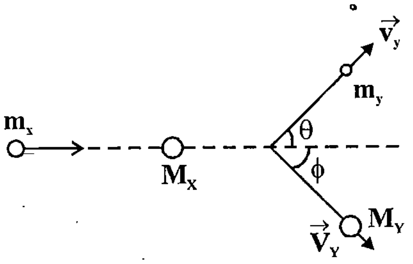
(i) Conservation of mass number
(ii) Conservation of atomic number
(iii) Conservation of energy (including mass-energy)
(iv) Conservation of linear momentum
(v) Conservation of angular momentum.
(vi) Conservation of parity
(vii) Conservation of isotopic spin

**Kinematics of nuclear reactions:**

Let a particle of mass $m_x$ moving along $x$-axis with a velocity $v_x$ collide elastically with a target nucleus of mass $M_X$ at rest. Due to the nuclear reaction, the product nucleus of mass $M_Y$ is scattered at an angle $\phi$ with velocity $v_Y$ and the product particle of $m_y$ is emitted with velocity $v_y$ at an angle $\theta$

From the law of conservation of linear momentum along $x$ and $y$ directions, we get

$$\left.\begin{aligned} M_Y u_Y \cos\phi + m_y v_y \cos\theta &= m_x v_x \\ -M_Y v_Y \sin\phi + m_y v_y \sin\theta &= 0 \end{aligned}\right\}$$



Or, $$\left.\begin{aligned} M_Y v_Y \cos\phi &= m_x v_x - m_y v_y \cos\theta \\ M_Y v_Y \sin\phi &= m_y v_y \sin\theta \end{aligned}\right\}$$

Squaring the above two equations and adding, we obtain

$$M_Y^2 v_Y^2 = m_x^2 v_x^2 + m_y^2 v_y^2 - 2m_x m_y v_x v_y \cos\theta$$

Or, $$2M_Y K_Y = 2m_x K_x + 2m_y K_y - 4\sqrt{m_x m_y K_x K_y}\cos\theta$$

since, non-relativistically, K = kinetic energy = $\frac{1}{2}mv^2$

$$\therefore \quad K_Y = \frac{m_x}{M_Y}K_x + \frac{m_y}{M_Y}K_y - \frac{2}{M_Y}\sqrt{m_x m_y K_x K_y}\cos\theta$$

But, $Q = K_Y + K_y - K_x$

$$\therefore \quad Q = K_y\left(1+\frac{m_y}{M_Y}\right) - K_x\left(1-\frac{m_x}{M_Y}\right) - \frac{2}{M_Y}\sqrt{m_x m_y K_x K_y}\cos\theta$$

This gives the Q-value of the reaction in terms of $K_y$, $K_x$ and $\theta$ without involving the kinetic energy of the recoil nucleus, $K_Y$ and the mass $M_x$ of target nucleus and is called the standard form of Q-equation.

If $\theta = 90°$, the Q-expression simplifies to : $Q = K_y\left(1+\frac{m_y}{M_Y}\right) - K_x\left(1-\frac{m_x}{M_Y}\right)$

If Q is negative, it gives the minimum value of threshold kinetic energy to be given to the incident particle for an endoergic reaction to proceed in the forward direction.

- **Q-value and threshold energy of nuclear reaction:**
  The law of conservation of energy and momentum imposes certain restrictions on the reactions. These restrictions are called the kinematic restrictions and this mathematical methods is known as kinematics. Consider the nuclear reaction

$$x + X \longrightarrow y + Y$$

Where x, X, y and Y are the bombarding particle, target nucleus, outgoing particle and product nucleus respectively. It is assumed that the target nucleus is in rest. Since total energy is conserved in the nuclear reaction, therefore we get,

$$\left(m_x c^2 + E_x\right) + M_X c^2 = \left(E_y + m_y c^2\right) + \left(E_Y + M_Y c^2\right)$$

$E_x, E_y$ and $E_Y$ are the kinetic energies of respective particles.

Now the quantity $Q = E_y + E_Y - E_x \Rightarrow Q = \left(m_x + M_X - m_y - M_Y\right)c^2$

Where Q is called the Q-value of nuclear reaction.

(i) If Q is positive, the reaction is said to be exoergic (exothermic) and
(ii) If Q is negative, the reaction is called endoergic (endothermic).

The minimum K.E. required for incident particle (x) to start the nuclear reaction is called the threshold energy $(E_x)^{th}$. The relation between Q-values and threshold energy is:

$$\boxed{E_x^{th} = -Q\frac{(m_y + M_Y)}{(M_Y + m_y - m_x)}}$$

If $E_x^{th} = 0$ for exoergic or exothermic reactions i.e. these reaction are spontaneous process.

- **Mechanism of nuclear reactions:**
- **Transmutation by protons:**

**(a) $(p,\alpha)$ reactions:**

(i) ${}_3Li^7 + {}_1H^1 \rightarrow ({}_4Be^8)^* \rightarrow {}_2He^4 + {}_2He^4$

(ii) ${}_3Li^6 + {}_1H^1 \rightarrow ({}_4Be^7)^* \rightarrow {}_2He^3 + {}_2He^4$

(iii) ${}_4Be^9 + {}_1H^1 \rightarrow ({}_5B^{10})^* \rightarrow {}_3Li^6 + {}_2He^4$

(iv) ${}_5B^{11} + {}_1H^1 \rightarrow ({}_6C^{12})^* \rightarrow {}_4Be^8 + {}_2He^4$

(v) ${}_9F^{19} + {}_1H^1 \rightarrow ({}_{10}Ne^{20})^* \rightarrow {}_8O^{16} + {}_2He^4$

**(b) (p–n ) Reaction:**

(i) ${}_5B^{11} + {}_1H^1 \rightarrow ({}_6C^{12})^* \rightarrow {}_6C^{11} + {}_0n^1$

(ii) ${}_8O^{18} + {}_1H^1 \rightarrow ({}_9F^{19})^* \rightarrow {}_9F^{18} + {}_0n^1$

(iii) ${}_{11}Na^{23} + {}_1H^1 \rightarrow ({}_{12}Mg^{24})^* \rightarrow {}_{12}Mg^{23} + {}_0n^1$

(iv) ${}_{28}Ni^{58} + {}_1H^1 \rightarrow ({}_{29}Cu^{59})^* \rightarrow {}_{29}Cu^{58} + {}_0n^1$

(v) ${}_{29}Cu^{65} + {}_1H^1 \rightarrow ({}_{30}Zn^{66})^* \rightarrow {}_{30}Zn^{65} + {}_0n^1$ (usually endoergic)

**(c) (p, d) reaction:**

(i) ${}_3Li^7 + {}_1H^1 \rightarrow {}_3Li^6 + {}_1H^2$

(ii) ${}_4Be^9 + {}_1H^1 \rightarrow {}_4Be^8 + {}_1H^2$

**(d) Proton capture:**

Compound nucleus in excited states come to ground state with $\gamma$–ray photon.

(i) ${}_3Li^7 + {}_1H^1 \rightarrow ({}_4Be^8)^* \rightarrow {}_4Be^8 + \gamma$

(ii) ${}_6C^{12} + {}_1H^1 \rightarrow ({}_7N^{13}) \rightarrow {}_7N^{13} + \gamma$

(iii) ${}_7N^{14} + {}_1H^1 \rightarrow ({}_8O^{15})^* \rightarrow {}_8O^{15} + \gamma$

(iv) ${}_9F^{19} + {}_1H^1 \rightarrow ({}_{10}Ne^{20})^* \rightarrow {}_{10}Ne^{20} + \gamma$

(v) ${}_{13}Al^{27} + {}_1H^1 \rightarrow ({}_{14}Si^{28})^* \rightarrow {}_{14}Si^{28} + \gamma$

(vi) ${}_{24}Cr^{50} + {}_1H^1 \rightarrow ({}_{25}Mn^{51})^* \rightarrow {}_{25}Mn^{51} + \gamma$

If incident proton has an energy (> 20 MeV) the compound nucleus has sufficient excitation energy to permit the expulsion of two or more nucleons.

**(2) Transmutation by Neutrons:-**

Neutrons have no electric charge and can penetrate positively charged nuclei without any experience of repulsive electrostatic force.

**(a) (n–$\alpha$) reaction:**

**With slow Neutrons**

(i) ${}_3Li^6 + {}_0n^1 \rightarrow ({}_3Li^7)^* \rightarrow {}_1H^3 + {}_2He^4$

(ii) ${}_5B^{10} + {}_0n^1 \rightarrow ({}_5B^{11})^* \rightarrow {}_3Li^7 + {}_2He^4$

(iii) ${}_{13}Al^{27} + {}_0n^1 \rightarrow ({}_{13}Al^{28})^* \rightarrow {}_{11}Na^{24} + {}_2He^4$

(iv) ${}_7N^{14} + {}_0n^1 \rightarrow ({}_7N^{15})$

(v) ${}_5B^{11} + {}_2He^4$ (vi) ${}_3Li^7 + {}_2He^4 + {}_2He^4$

Capture of fast Neutrons - emission of $\alpha$ - particle are usually radioactive

(i) ${}_{11}Na^{23} + {}_0n^1 \rightarrow \left({}_{11}Na^{24}\right)^* \rightarrow {}_9F^{20} + {}_2He^4$ followed by ${}_9F^{20} \rightarrow {}_{10}Ne^{20} + \beta^-$

(ii) ${}_{13}Al^{27} + {}_0n^1 \rightarrow \left({}_{13}Al^{28}\right)^* \rightarrow {}_{11}Na^{24} + {}_2He^4$ followed by ${}_{11}Na^{24} \rightarrow {}_{12}Mg^{24} + \beta^-$

**(b) (n, p) reaction:** Proton in the nucleus is replaced by neutron mass number, does not change but change decreases by one unit.

(i) ${}_7N^{14} + {}_0n^1 \rightarrow \left({}_7N^{15}\right)^* \rightarrow {}_6C^{14} + {}_1H^1 + Q$ followed by ${}_6C^{14} \rightarrow {}_7N^{14} + \beta^- + E_{max}$

(ii) ${}_2He^3 + {}_0n^1 \rightarrow \left({}_2He^4\right)^* \rightarrow {}_1H^3 + {}_1H^1 + Q$ followed by ${}_1H^3 \rightarrow {}_2He^3 + \beta^- + E_{max}$

(iii) ${}_{17}Cl^{35} + {}_0n^1 \rightarrow \left({}_{17}Cl^{36}\right)^* \rightarrow {}_{16}S^{35} + {}_1H^1 + Q$ followed by ${}_{16}S^{35} \rightarrow {}_{17}Cl^{35} + \beta^- + E_{max}$

**With fast neutrons:**

**(i)** ${}_{13}Al^{27} + {}_0n^1 \rightarrow \left({}_{13}Al^{28}\right)^* \rightarrow {}_{12}Mg^{27} + {}_1H^1$ **(ii)** ${}_{16}S^{32} + {}_0n^1 \rightarrow \left({}_{16}S^{33}\right)^* \rightarrow {}_{15}P^{32} + {}_1H^1$

**(iii)** ${}_{29}Cu^{65} + {}_0n^1 \rightarrow \left({}_{29}Cu^{66}\right)^* \rightarrow {}_{28}Ni^{65} + {}_1H^1$ **(iv)** ${}_{30}Zn^{64} + {}_0n^1 \rightarrow \left({}_{30}Zn^{65}\right)^* \rightarrow {}_{29}Cu^{64} + {}_1H^1$

**(c)** $(n-\gamma)$ **Reaction:**

**(i)** ${}_1H^1 + {}_0n^1 \rightarrow \left({}_1H^2\right)^* \rightarrow {}_1H^2 + \gamma$ **(ii)** ${}_1H^2 + {}_0n^1 \rightarrow \left({}_1H^3\right)^* \rightarrow {}_1H^3 + \gamma$

**Product Nucleus is Radioactive:**

**(i)** ${}_{45}Rh^{103} + {}_0n^1 \rightarrow \left({}_{45}Rh^{104}\right)^* \rightarrow {}_{45}Rh^{104} + \gamma$ **(ii)** ${}_{49}In^{115} + {}_0n^1 \rightarrow \left({}_{49}In^{116}\right)^* \rightarrow {}_{49}In^{116} + \gamma$

**(iii)** ${}_{92}U^{238} + {}_0n^1 \rightarrow \left({}_{92}U^{239}\right)^* \rightarrow {}_{92}U^{239} + \gamma$ **(iv)** ${}_{29}Cu^{65} + {}_0n^1 \rightarrow \left({}_{29}Cu^{66}\right)^* \rightarrow {}_{29}Cu^{66} + \gamma$

**(v)** ${}_{79}Au^{65} + {}_0n^1 \rightarrow \left({}_{79}Au^{198}\right)^* \rightarrow {}_{79}Au^{198} + \gamma$

**(d) (n – d ) and (n – t) reaction:** Bombardment of 90 MeV neutrons: ${}_7N^{14} + {}_0n^1 \rightarrow {}_6C^{12} + {}_1H^3$

**(e) (n – 2n ) reaction:** One neutron is captured by nucleus and 2 neutrons are emitted. $Q < 0$ fast neutron are needed . In most cases, residue nucleus is unstable -followed by positron emission.

(i) ${}_6C^{11} + {}_0n^1 \rightarrow \left({}_6C^{13}\right)^* \rightarrow {}_6C^{11} + 2\,{}_0n^1$ (ii) ${}_{19}K^{39} + {}_0n^1 \rightarrow \left({}_{19}K^{40}\right)^* \rightarrow {}_{19}K^{38} + 2\,{}_0n^1$

(iii) ${}_{51}Sb^{121} + {}_0n^1 \rightarrow \left({}_{51}Sb^{122}\right)^* \rightarrow {}_{51}Sb^{120} + 2\,{}_0n^1$

**(f) Neutron - Three or more particles:** Incident neutron ~ 30 MeV sufficient energy to overcome Coulomb barrier, 3 neutrons or even 2 neutrons and a proton are ejected from compound nucleus. Neutron (~ 100 MeV) Nuclei with moderate mass number undergo spallation and those of high mass number, e.g. Bi and Pb suffer fission probably accompanied by spallation.

- **(3) Transmutation by Deuteron:** High energy Deuteron.

**(a) $(d-\alpha)$ reaction**

(i) ${}_3Li^6 + {}_1H^2 \rightarrow \left({}_4Be^8\right)^* \rightarrow {}_2He^4 + {}_2He^4$  (ii) ${}_8O^{16} + {}_1H^2 \rightarrow \left({}_9F^{18}\right)^* \rightarrow {}_7N^{14} + {}_2He^4$,

(iii) ${}_{10}Ne^{20} + {}_1H^2 \rightarrow \left({}_{11}Na^{22}\right)^* \rightarrow {}_9F^{18} + {}_2He^4$

(iv) ${}_{12}Mg^{26} + {}_1H^2 \rightarrow \left({}_{13}Al^{28}\right)^* \rightarrow {}_{11}Na^{24} + {}_2He^4$,

(v) ${}_{13}Al^{27} + {}_1H^2 \rightarrow \left({}_{14}Si^{29}\right)^* \rightarrow {}_{12}Mg^{25} + {}_2He^4$

**(b) (d – p) Reaction: Isotope creation**

(i) ${}_6C^{12} + {}_1H^2 \rightarrow \left({}_7N^{14}\right)^* \rightarrow {}_6C^{13} + {}_1H^1$  (ii) ${}_3Li^7 + {}_1H^2 \rightarrow \left({}_4Be^9\right)^* \rightarrow {}_3Li^8 + {}_1H^1$

(iii) ${}_{11}Na^{23} + {}_1H^2 \rightarrow \left({}_{12}Mg^{25}\right)^* \rightarrow {}_{11}Na^{24} + {}_1H^1$  (iv) ${}_{15}P^{31} + {}_1H^2 \rightarrow \left({}_{16}S^{33}\right)^* \rightarrow {}_{15}P^{32} + {}_1H^1$

(v) ${}_{48}Cd^{114} + {}_1H^2 \rightarrow \left({}_{49}In^{116}\right)^* \rightarrow {}_{48}Cd^{115} + {}_1H^1$

(vi) ${}_{83}Bi^{209} + {}_1H^2 \rightarrow \left({}_{84}Po^{211}\right)^* \rightarrow {}_{83}Bi^{210} + {}_1H^1$

**(c) (d – n) Reaction:**

(i) ${}_3Li^7 + {}_1H^2 \rightarrow \left({}_4Be^9\right)^* \rightarrow {}_4Be^8 + {}_0n^1$  (ii) ${}_4Be^9 + {}_1H^2 \rightarrow \left({}_5B^{11}\right)^* \rightarrow {}_5B^{10} + {}_0n^1$

(iii) ${}_6C^{12} + {}_1H^2 \rightarrow \left({}_7N^{14}\right)^* \rightarrow {}_7N^{13} + {}_0n^1$

When two deuterons interact both the (d, n) and (d, p) reactions have been obtained.

$$
\begin{aligned}
{}_1H^2 + {}_1H^2 \rightarrow {}_2He^4 &\rightarrow {}_1H^3 + {}_1H^1 + 4.02\ \text{MeV} \\
&\rightarrow {}_2He^3 + {}_0n^1 + 3.25\ \text{MeV}
\end{aligned}
$$

- **Cross-section of nuclear reactions (Classical):**

One of the most important parameters in nuclear reactions is the reaction cross-section, symbolised by $\sigma$. It is quantitative measure of the probability of occurrence of a nuclear reaction.

Let a parallel beam of $N$ monoenergetic particles be incident per unit time normally on a target foil of surface area $A$ and thickness $\Delta x$, having 'n' nuclei per unit volume. Now, the number $\Delta N$ of nuclei in the foil undergoing nuclear reaction will be proportional to (i) the intensity of the beam and (ii) the number of target nuclei contained in the the foil.

But the intensity $I$ of the beam $= N/A$ and the number of nuclei in the foil $= nA.\Delta x$

Therefore, number of nuclei transmuted is $\Delta N \propto \frac{N}{A} nA\Delta x$

$\therefore\ \Delta N = \sigma N n \Delta x = \sigma N n_1$, where $\sigma$ = constant.

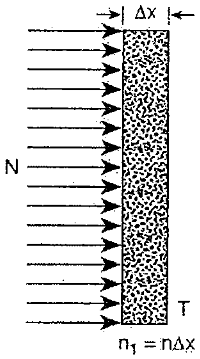


**Figure:** Nuclear reaction cross-section: bombardment of target foil.

where $n_1 = n\Delta x$, the number of nuclei per unit area of the target foil used.

The constant $\sigma$ is called the nuclear reaction cross-section.

Therefore, nuclear reaction cross-section, $\boxed{\sigma = \frac{\Delta N}{N n_1}}$

- The probability of the incident particle to interact with the target atom will be equal to

$$P = \frac{\text{number of interacting particles}}{\text{number of incident particles}}$$

If $dN$ is number of interacting particles and $N$ is number of incident particles, then

$$P = \boxed{\frac{dN}{N} = \sigma\, n\, dx} \Rightarrow \int_{N_0}^{N} \frac{dN}{N} = \int_{z=z}^{z=0} \sigma\, n\, dx \Rightarrow \ln\left[\frac{N}{N_0}\right] = -\sigma\, n(+x) \Rightarrow \frac{N}{N_0} = e^{-\sigma\, nx}$$

$$\boxed{N = N_0\, e^{-\sigma\, n\, x}}$$

## 8.6. Nuclear Shell Model:

1. It very well explains the existence of magic numbers and the stability and high binding energy on the basis of closed shells.
2. The shell model provides explanation for the ground state spins and magnetic dipole moment of the nuclei. The neutrons and protons with opposite spins pair off so that the mechanical and magnetic dipole moment cancel and the odd or left out proton or neutron contributes to the spin and magnetic moment of the nuclei as a whole.
3. Nuclear isomerism, i.e., existence of isobaric, isotopic nuclei in different energy states of odd-$A$ nuclei between 39-49, 69-81, 111 to 125 has been explained by shell model by the large difference in nuclear spins of isomeric states are their $A$-values are close to magic numbers.

**Application of Nuclear Shell Model:**

**Determination of total angular momentum and parity of the ground state**

1. If protons and neutrons both are even, then the total angular momentum become zero and parity is even.
2. If one of the nucleon (proton or neutron) is even and other is odd then the $j$ value and $(-1)^{\ell}$ of the last unpaired nucleon give the total angular momentum and parity.
3. For odd-odd nucleon we use Brennan-Bernstein rules. According to this rules we define

the quantity, $N=(j_P-\ell_P)+(j_N-\ell_N)$

(i) If $N=0$, then $j=|j_P-j_N|$, parity $=-ve$

(ii) If $N=\pm1$ and $N=Z$ then $J=|J_P+J_N|$, parity $=\pm1$

(iii) If $N=\pm1$ and $N\neq Z$ then $J=(J_P+J_N-1)$, parity $=+ve$

**Example :** ${}_8O^{16}$

$$Z=8\text{ (even) }\rightarrow 1s_{1/2}^2\,1p_{3/2}^4\,1p_{1/2}^2$$

$$N=8\text{ (even) }\rightarrow 1s_{1/2}^2\,1p_{3/2}^4\,1p_{1/2}^2$$

Since, there have no unpaired nucleon.
Therefore, parity is positive and angular momentum $j = 0$. It is represented by $0^+$.

${}^{27}_{13}Al$ : $Z=13(\text{odd})\rightarrow 1s_{1/2}^2\,1p_{3/2}^4\,1p_{1/2}^2\,1d_{5/2}^5$

$N=14(\text{even})\rightarrow$ does not contribute in angular momentum

Therefor, $j = 5/2$; $l = 2$ (for d, $l = z$)

Parity = $(-1)^2 = +1$, even parity. It is represented by $\frac{5}{2}^+$.

${}^{33}_{16}S$ : $Z=16(\text{even})\rightarrow$ does not contribute in angular momentum

$N=17(\text{odd})\rightarrow 1s_{1/2}^2\,1p_{3/2}^4\,1p_{1/2}^2\,1d_{5/2}^5\,2s_{1/2}^2\,1d_{3/2}^2$

Therefor, $j = 3/2$; $l = 2$ (for d, $l = 2$)

Parity = $(-1)^2 = +1$, even parity. It is represented by $\frac{3}{2}^+$.

${}^{38}Cl_{17}$:

$Z=17(\text{odd})\rightarrow 1s_{1/2}^2\,1p_{3/2}^4\,1p_{1/2}^2\,1d_{5/2}^6\,2s_{1/2}^2\,1d_{3/2}^1$ (particle state)

$N=21(\text{odd})\rightarrow 1s_{1/2}^2\,1p_{3/2}^4\,1p_{1/2}^2\,1d_{5/2}^6\,2s_{1/2}^2\,1d_{3/2}^4\,1f_{7/2}^1$ (particle state)

$$N=(j_P-\ell_P)+(j_N-\ell_N)=-\frac{1}{2}+\frac{1}{2}=0$$

$$\therefore\quad j=|j_P-j_N|=\left|\frac{3}{2}-\frac{7}{2}\right|=\frac{4}{2}=2\text{ and parity}=-1$$

Therefore, spin parity = $2^-$.

${}^{26}Al_{13}$:

$Z=13(odd)=1s_{1/2}^2\,1p_{3/2}^4\,1p_{1/2}^2\,1d_{5/2}^5$

$N=13(odd)=1s_{1/2}^2\,1p_{3/2}^4\,1p_{1/2}^2\,1d_{5/2}^5$

$$N=(J_P-\ell_P)+(J_N-\ell_N)=\frac{1}{2}+\frac{1}{2}=1\quad\text{and } N=Z$$

$$J=|J_P+J_N|=\left|\frac{5}{2}+\frac{5}{2}\right|=5,\text{ parity}=+\text{ve}\quad\text{and parity is +ve}$$

Therefore, spin parity $5^+$.

$^{56}Co_{27}$:

$Z = 27(odd) = 1s_{1/2}^2\ 1p_{3/2}^4\ 1p_{1/2}^2\ 1d_{5/2}^6\ 2s_{1/2}^2\ 1d_{3/2}^4\ 1f_{7/2}^7$

(hole state, because only one particle state is empty)

$N = 29(odd) = 1s_{1/2}^2\ 1p_{3/2}^4\ 1p_{1/2}^2\ 1d_{5/2}^6\ 2s_{1/2}^2\ 1d_{3/2}^4\ 1f_{7/2}^8\ 2p_{3/2}^1$ (particle state)

$$N = (j_P - \ell_P) + (j_N - \ell_N) = \frac{1}{2} + \frac{1}{2} = 1 \text{ and } N \neq Z$$

$$\therefore \quad J = |j_P + j_N - 1| = \frac{7}{2} + \frac{3}{2} - 1 = 4, \text{ parity} = +ve$$

Therefore, spin parity = $4^+$.

## 8.7. Ground State of Deuteron $\left({}_1D^2\right)$

Deuteron is a single two nucleon (one *p* and one *n*) bound system which is found in nature. The important experimental determinal properties about Deuteron are given below:

(i) The *B.E.* of Deuteron/nucleon is very small compare to other nuclei i.e. it is a weakly system.

(ii) The ground state spin of Deuteron $I_d = 1$ (iii) The ground state parity of Deuteron = even (positive)

(iv) The quadrupole momento of Deuteron $(Q_d) \neq 0$

(v) The magnetic moment of Deuteron is slightly different from the sum of intrinsic magnetic moments of neutron + proton i.e. $(\mu_n + \mu_p) - \mu_d \to 0$

These factor represents that the ground state of Deuteron is a mixture of $\underset{(L=0)}{{}^3S}$ and $\underset{(L=2)}{{}^3D}$ states in which ${}^3S$ contribution is 96% and ${}^3D$ contribution is only 4%.

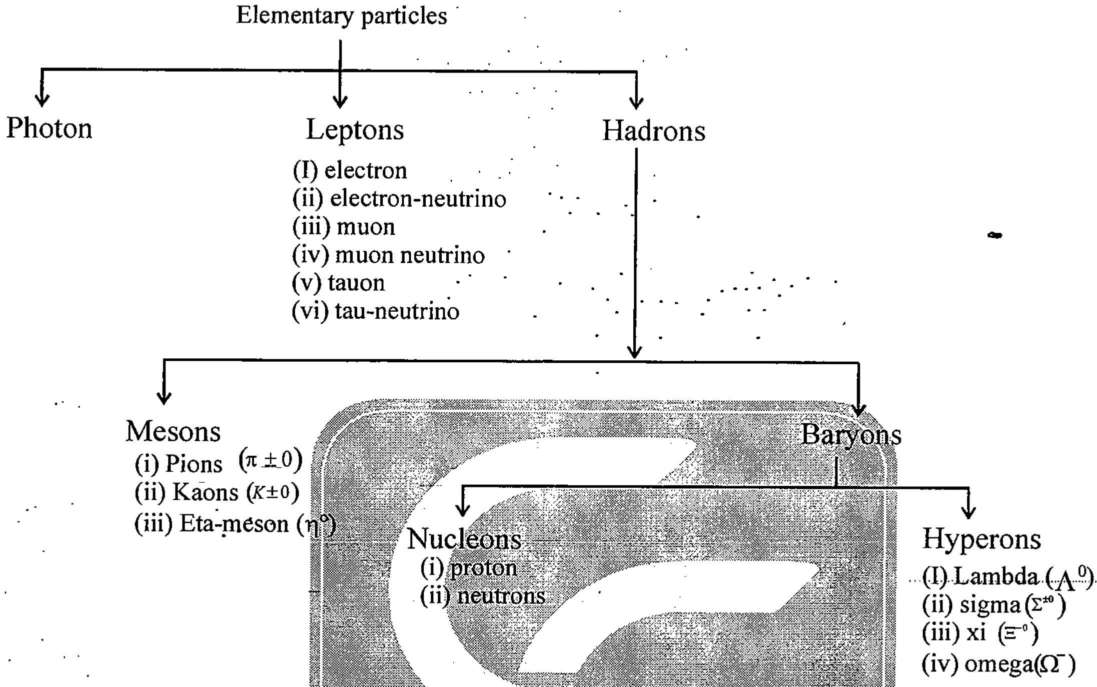
This indicate that nuclear force are non-central as $L \neq$ constant and they are spin dependent. In Deuteron both *p* and *n* have spin parallel to each other.

## 8.8. Elementary Particles:

It is a dream for physicists to be able to explain the material world, the physical universe entirely. This has driven them continuously to explore the fundamental entities of matter was atom then nucleus, neutron and protons and now the quarks. In this present chapter we will read about all elementary particle their spin charge, isospin, mass etc.

**Classification of elementary particles:**

The elementary particles may be classified in a number of different ways depending on their masses, interaction, statistics etc. Commonly, however, they are classified into such categories as


Elementary particles
Photon
Leptons
(I) electron
(ii) electron-neutrino
(iii) muon
(iv) muon neutrino
(v) tauon
(vi) tau-neutrino
Hadrons
Mesons
(i) Pions ($\pi \pm 0$)
(ii) Kaons ($K\pm 0$)
(iii) Eta-meson ($\eta^{\circ}$)
Baryons
Nucleons
(i) proton
(ii) neutrons
Hyperons
(I) Lambda ($\Lambda^0$)
(ii) sigma ($\Sigma^{\pm 0}$)
(iii) xi ($\Xi^{-0}$)
(iv) omega($\Omega^-$)

**The four fundamental forces:**

In nature, there are four different types of fundamental interactions. They are:

1. **Gravitational force :**
   The gravitational force is the oldest known force. It holds us to the surface of the earth and holds the entire universe together. It is a long-range force varying as $\frac{1}{r^2}$. It has life time, $\tau = 10^{16}$ sec.
2. **Electromagnetic force:**
   The electromagnetic force was the second force known. In fact it was originally two forces the electric force and magnetic force. The electromagnetic force hold the atoms, molecules, solid and liquid together. It is also a long range force varying as $\frac{1}{r^2}$. It has a life time, $\tau = 10^{-20}$ sec.
3. **Weak interaction force:**
   This fundamental interaction involves leptons and hadrons. The $\beta$-decay of radioactive nuclei and decays of strong particles are typically of weak interaction. It has a life time, $\tau = 10^{-10}$ sec.
4. **Strong interaction force :**
   The strong nuclear force is responsible for holding the nucleons together. It is the strongest of all the forces but is a very short range force. Its effect occur within a distance of about $10^{-15}$ m. The strong nuclear force acts only on the hadrons. It has a life time, $\tau = 10^{-23}$ sec.

**Some elementary particle and their properties:**

| Category | Particles Name | Symbols | Charge | Spin | B | L | s | $I_3$ | Y | Quark structure |
|---|---|---|---|---|---|---|---|---|---|---|
|  | Electron | $e^-$ | –1 | ½ | 0 | 1 | 0 |  | 0 |  |
|  | Electron-neutrino | $\nu_e$ | 0 | ½ | 0 | 1 | 0 |  | 0 |  |
| Leptons | Muon | $\mu^-$ | –1 | ½ | 0 | 1 | 0 |  | 0 |  |
|  | Muon-Neutrino | $\nu_\mu$ | 0 | ½ | 0 | 1 | 0 |  | 0 |  |
|  | Tauon | $\tau$ | –1 | ½ | 0 | 1 | 0 |  | 0 |  |
|  | Tauon-Neutrino | $\nu_\tau$ | 0 | ½ | 0 | 1 | 0 |  | 0 |  |
|  |  | $\pi^+$ | +1 | 0 | 0 | 0 | 0 | +1 | 0 | $u\bar{d}$ |
|  | Pions | $\pi^0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{1}{\sqrt{2}}(u\bar{u}-d\bar{d})$ |
| Mesons |  | $\pi^-$ | –1 | 0 | 0 | 0 | 0 | –1 | 0 | $d\bar{u}$ |
|  |  | $K^+$ | +1 | 0 | 0 | 0 | +1 | -½ | +1 | $u\bar{s}$ |
|  | Kaons | $K^0$ | 0 | 0 | 0 | 0 | +1 | -½ | +1 | $d\bar{s}$ |
|  |  | $K^-$ | –1 | 0 | 0 | 0 | –1 | ½ | –1 | $s\bar{u}$ |
|  | Eta | $\eta^0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{1}{\sqrt{6}}(u\bar{u}+d\bar{d}-2s\bar{s})$ |
|  | $\eta^1$ | $\eta'$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $(u\bar{u}+d\bar{d}+s\bar{s})\frac{1}{\sqrt{3}}$ |
| Nucleons | Proton | P | +1 | ½ | 1 | 0 | 0 | ½ | 1 | uud |
|  | Neutron | n | 0 | ½ | 1 | 0 | 0 | -½ | 1 | udd |
| Hyperons | Lambda | $\Lambda^0$ | 0 | ½ | 1 | 0 | –1 | ½ | 0 | uds |
|  | Sigma | $\Sigma^+$ | +1 | ½ | 1 | 0 | –1 | +1 | 0 | uus |
|  |  | $\Sigma^0$ | 0 | ½ | 1 | 0 | –1 | 0 | 0 | uds |
|  |  | $\Sigma^-$ | –1 | ½ | 1 | 0 | –1 | 0 | 0 | dds |
|  | Xi | $\Xi^0$ | 0 | ½ | 1 | 0 | -2 | ½ | –1 | uss |
|  |  | $\Xi^-$ | –1 | ½ | 1 | 0 | -2 | -½ | -1 | dss |
|  | Omega | $\Omega^-$ | -1 | ½ | 1 | 0 | –3 | 0 | –2 | sss |

**Conservation laws in elementary particle reactions:**

- **Exact conservation laws:**
  (i) Conservation of linear momentum
  (ii) Conservation of angular momentum
  (iii) Conservation of charge
  (iv) Conservation of baryon number
  (v) Conservation of lepton number
- **Approximate conservation laws:**
  **(i) Isospin or isotopic spin($I$):**
  It arose however from the idea that pairs of particles like nucleons and triplets like pions hardly differ in their mass and may be considered as isotopes and that their charges, differing from each other by unity, suggest space qunatization similar to electron spin and orbit in a magnetic field.
- **A multiplet number** ($M$) is defined as the number of their different charge states. For instance, for nucleons i.e. protons or neutrons - the multiplet number $M = 2$. Similarly, for the triplet of pions, $M = 3$; $M = 2$ for two kaons and two antikaons; $M = 1$ for one neutral $\eta^0$.

  Now, $M = 2I+1 \quad \Rightarrow I = (M-1)/2$

- Isospin is treated as a vector $\vec{I}$ of magnitude $\sqrt{I(I+1)}$, like angular momentum, but $I$ is dimensionless. Its component along $Z$-axis, is given by $I_3$ which have the allowed values.

  $I, (I-1), (I-2), \ldots\ldots\ldots\ldots -I$

- For nucleons $(M=2), I=(M-1)/2=\frac{1}{2}$ and the values of $I_3$ are $+\frac{1}{2}$ and $-\frac{1}{2}$; $I_3=+\frac{1}{2}$ is assigned to proton, $I_3=-\frac{1}{2}$ to neutron.

- For pions $(M=3), I=(M-1)/2=1$. Hence, $I_3=+1, 0, -1$. $I_3=+1$ is assigned to $\pi^+$, $I_3=0$ to $\pi^0$ and $I_3=-1$ to $\pi^-$.

- Isospin is conserved in strong interactions but is violated in electromagnetic and weak interaction. The $Z$-component of isospin, $I_3$- is conserved in strong and electromagnetic interaction and not in weak interaction.

**(ii) Hypercharge($Y$):** It is defined as double of the average charge $\bar{Q}$ of the multiplet.

$$Y=2\bar{Q} \Rightarrow \bar{Q}=Y/2 \quad \Rightarrow \boxed{Q=I_3+Y/2}$$

- For any strong and electromagnetic interaction, the hypercharge is conserved, i.e. remains invariant. But it need not be conserved in weak interaction.

**Example:** $p+p \to \Lambda^0+K^0+p+\pi^+$

Hypercharge: $1+1=0+1+1+0 \quad \Rightarrow \Delta Y=0$

**(iii) Strangeness number ($S$):** It is defined as the difference of the hyperchange $Y$ and the baryon number $B$.

$$S=Y-B \quad \Rightarrow Y=S+B$$

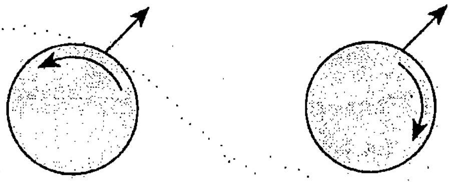
i.e., the hyperchange is the sum of the baryon number and the strangeness number

Therefore, $Q=I_3+\frac{B+S}{2}$

- Strangeness number S is conserved in strong and electromagnetic reactions. For a weak interactions, $\Delta S=0$ or $\pm 1$

**(iv) Parity:**

When particle like neutrinos are emitted during radioactive decay, they show a preferred spin direction. If a neutrino spins in the direction at which a right-handed screw advances, it is said to posses a helicity +1; if however the spin is in the direction of a left-handed screw, the helicity is $-1$. As the parity $P$ is related to the spin $J$, the two quantum numbers are usually combined and is symbolised by $J^P$. So, $\left(\frac{1}{2}\right)^+$ means the $J$-value is $\frac{1}{2}$ and $P=+1$; $\left(\frac{1}{2}\right)^-$ indicates $J=\frac{1}{2}, P=-1$.



Neutrino
(a)

Anti-Neutrino
(b)

**Figure:** Helicity of neutrino and antineutrino

- In strong and electromagnetic interactions, parity is conserved but it is not conserved in weak interaction.

**(v) Charge conjugation:**

Charge conjugation means reversal of the signs of all types of charge i.e. electric, baryonic and leptonic of the particles (Figure below). If a physical law holding for particles also holds for corresponding antiparticles, the principle of charge conjugation is said to be valid.

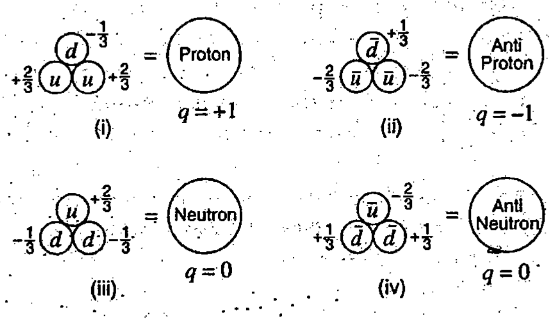

$d$
$-\frac{1}{3}$
$+\frac{2}{3}$ $u$ $u$ $+\frac{2}{3}$
Proton
$q=+1$
(i)
$\bar{d}$
$+\frac{1}{3}$
$-\frac{2}{3}$ $\bar{u}$ $\bar{u}$ $-\frac{2}{3}$
Anti Proton
$q=-1$
(ii)
$u$
$+\frac{2}{3}$
$-\frac{1}{3}$ $d$ $d$ $-\frac{1}{3}$
Neutron
$q=0$
(iii)
$\bar{u}$
$-\frac{2}{3}$
$+\frac{1}{3}$ $\bar{d}$ $\bar{d}$ $+\frac{1}{3}$
Anti Neutron
$q=0$
(iv)

**Figure:** Formation of proton, antiproton and neutron, antineutron by charge conjugation

- Strong and electromagnetic interactions are charge conjugate invariant. But the weak interaction like $\beta$-decay does not obey charge conjugation.

**(vi) Time reversal:**

The operation $T$ i.e., time-reversal means replacing the time $t$ by $-t$ in all equations of motion i.e., reflection of time axis at the origin of time coordinate in relativistic space-time continuum. It is thus, like the parity operation, a discrete change.

- $T$-operation consists of reversing the signs of momenta $\left(\vec{p} = d\vec{r}/dt\right)$ and angular momenta $\left(\vec{L} = \vec{r}\times\vec{p}\right)$.

  $T$ also transforms the wave function to its complex conjugate. If $T$ is conserved i.e., time-reversal invariance occurs, then the reversed equation of motion is also a valid equation of motion of the system concerned. All the known fundamental equations of motion are invariant in time-reversal.
- Strong and electromagnetic interactions are invariant under time-reversal transformation.

***CPT* theorem:**

This is an exact conservation law. It states that all interactions in nature are invariant under joint operations of charge conjugation ($C$), inversion of space coordinates at origin, i.e. parity ($P$) and reversal of time ($T$). The order of operations is immaterial.

The invariance of $CPT$ transformation implies that if any interaction is not invariant under any one of $C$, $P$ and $T$ operations, its effect gets compensated by the joint effect of the other two.

**Conserved quantities in different type of reactions:**

| Conserved Quantities | Strong | Electromagnetic | Weak |
|---|---|---|---|
| 1. Charge | ✓ | ✓ | ✓ |
| 2. Linear Momentum | ✓ | ✓ | ✓ |
| 3. Relativistic Energy | ✓ | ✓ | ✓ |
| 4. Spin | ✓ | ✓ | ✓ |
| 5. $L_e$ | ✓ | ✓ | ✓ |
| 6. $L_\mu$ | ✓ | ✓ | ✓ |
| 7. $L_\tau$ | ✓ | ✓ | ✓ |
| 8. Baryon Number B | ✓ | ✓ | ✓ |
| 9. $I_3$ | ✓ | ✓ | × |
| 10. I | ✓ | May or may not be conserved | × |
| 11. S | ✓ | ✓ | May or may not be conserved |
| 12. Y = B + S | ✓ | ✓ | May or may not be conserved |
| 13. Parity (P) | ✓ | ✓ | × |
| 14. Charge conjugation (C) | ✓ | ✓ | × |
| 15. Time reversion (T) | ✓ | May or may not be conserved | May or may not be conserved |
| 16. CP | ✓ | ✓ | May or may not be conserved |
| 17. CPT | ✓ | ✓ | ✓ |

**Relationship between particles and antiparticles:**

(a) Mass → Same
(b) Spin → Same
(c) Charge → Same but opposite in sign.
(d) Mag. moment → Same in magnitude but opposite in sign.
(e) Mean life time of free decay → Same
(f) Annihilation → In pairs
(g) Creation → In pairs
(h) Total isotopic spin → same
(i) Third component of isospin($I_3$) → Same in mag. but opposite in sign.
(j) Intrinsic parity → Same for bosons and opposite for Fermions.
(k) Strangeness number ($S$) → Same in mag. but opposite in sign.
(l) Lepton number ($L$) → Same in mag. but opposite in sign.
(m) Baryon number ($B$) → Same in mag. but opposite in sign.
(n) Hypercharge ($Y$) → Same in mag. but opposite in sign.

The particle and antiparticle have same symbol except a bar over particle for antiparticle.

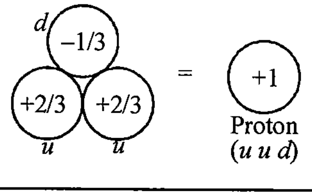
**Quark model:**

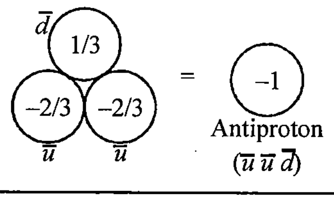
The elementary particles can be conceived (as far as isospin and hyper charge are concerned) as being built out of combination of quarks. The original three quarks were called up ($u$), down ($d$) and strange ($s$).

Each quarks has an anti-quark associated with it $(\bar{u}, \bar{d}, \bar{s})$. Since a quark and its anti-quark have opposite quantum numbers. They can be created from energy. In the reverse process, a quark and its anti-quark annihilate and give energy.

Thus, energy $\rightarrow u + \bar{u}$ ; $d + \bar{d} \rightarrow$ energy

Quark models for proton and antiproton.





## SOLVED PROBLEMS

1. If an electron is confined within a nucleus whose diameter is $10^{-14}$ m, estimate its minimum kinetic energy.

**Soln.** The de-Broglie wavelength of a minimum-energy electron confined inside the nucleus would be approximately twice the nuclear diameter (one-half a wavelength would fit into the diameter).

Therefore, the electron's momentum would be of the order of magnitude.

$$p = \frac{h}{\lambda} = \frac{hc}{\lambda c} = \frac{12.4\times 10^3\ \text{eV Å}}{\left(2\times 10^{-4}\ \text{Å}\right)c} = 62\times 10^6\ \frac{\text{eV}}{c} = 62\ \frac{\text{MeV}}{c}$$

corresponding to a kinetic energy of

$$K = \sqrt{(pc)^2 + E_0^2} - E_0 = \sqrt{\left(62\frac{\text{MeV}}{c}\times c\right)^2 - (0.511\ \text{MeV})^2} - 0.511\ \text{MeV} = 61\ \text{MeV}$$

2. Determine the radii of a $^{16}$O and a $^{208}$Pb nucleus.

**Soln.** From $R = r_0 A^{1/3} = (1.4\ \text{fm})A^{1/3}$,

$$R_0 = (1.4\ \text{fm})(16)^{1/3} = 3.53\ \text{fm};\ R_{Pb} = (1.4\ \text{fm})(208)^{1/3} = 8.29\ \text{fm}$$

3. Determine the stable nucleus that has a radius 1/3 that of $^{189}$Os.

**Soln.** Since, $R \propto A^{1/3}$

$$\frac{1}{3} = \frac{R}{R_{Os}} = \left(\frac{A}{A_{Os}}\right)^{1/3} = \left(\frac{A}{189}\right)^{1/3}$$

giving,

$$A = \frac{189}{27} = 7 \text{ corresponding to } ^7\text{Li.}$$

4. Calculate the binding energy of $^{126}_{52}Te$.

**Soln.** The binding energy is given by

$$BE = \left(Zm_p\right)c^2 + \left(Nm_n\right)c^2 - M_{\text{nucleus}}c^2$$

$$= (52\times 1.007\,825u + 74\times 1.008\,665\,u - 125.903\,322u)\times 931.5\ \text{MeV}/u = 1.066\times 10^3\ \text{MeV}$$

5. What is the energy required to remove the least tightly bound neutron from $^{40}_{20}Ca$?

**Soln.** From conservation of mass-energy,

$$M_{^{40}Ca}c^2 + E = \left(M_{^{39}Ca} + m_n\right)c^2$$

$$(39.962589u)(931.5\ \text{MeV}/u) + E = (38.970691\,u + 1.008665\,u)(931.5\ \text{MeV}/u)$$

$$\Rightarrow \quad E = 15.6\ \text{MeV}$$

6. Determine the electrical potential energy of the protons in a nucleus if it is assumed that the charge is uniformly speherically distributed.

**Soln.** Consider a thin spherical shell of charge, $dq = \rho PV = \rho\left(4\pi r^2 dr\right)$ that is added to a sphere that has the same charge density and has total charge

$$q = \rho V = \rho\left(\frac{4}{3}\pi r^3\right)$$

The electrical potential energy $dE$ of the thin shell is then

$$dE = \frac{kq}{r}dq = \frac{k}{r}\left(\frac{4}{3}\rho\pi r^3\right)\left(4\pi\rho r^2 dr\right) = 3k\left(\frac{4}{3}\rho\pi\right)^2 r^4 dr$$

The total electrical potential energy of the charged sphere is found by integrating $dE$ from $r = 0$ to $r = R$, the radius of the sphere

$$E = \int_0^R dE = 3k\left(\frac{4}{3}\rho\pi\right)^2 \int_0^R r^4 dr = \frac{3}{5}k\left(\frac{4}{3}\rho\pi\right)^2 R^5 = \frac{3k}{5R}\left(\frac{4}{3}\rho\pi R^3\right)^2$$

Since $\frac{4}{3}\rho\pi R^3 = \rho V = Q = Ze$, we have

$$E = \frac{3}{5}\frac{kZ^2e^2}{R}$$

The charges forming a nucleus are actually not continuous but must be brought in discrete amounts. For $Z = 1$ the Coulomb energy should be zero, but the above expression gives a finite answer. To correct the above relationship, $Z^2$ should be changed to $Z(Z-1)$. For large values of $Z$ this is a minor correction, but not for small values of $Z$. The correct Coulomb energy is:

$$E_C = \frac{3}{5}\frac{kZ(Z-1)e^2}{R}$$

7. What is the activity of one gram of ${}^{226}_{88}Ra$, whose half-life is 1622 years?

**Soln.** The number of atoms in 1 g of radius is

$$N(1\,g)\left(\frac{1\text{ g-mole}}{226\text{ g}}\right)\left(6.025\times10^{23}\ \frac{\text{atoms}}{\text{g-mole}}\right) = 2.66\times10^{21}$$

The decay constant is related to the half-life by

$$\lambda = \frac{0.693}{T_{1/2}} = \left(\frac{0.693}{1622\ y}\right)\left(\frac{1\ y}{365\ d}\right)\left(\frac{1\ d}{8.64\times10^4\ s}\right) = 1.355\times10^{-11}s^{-1}$$

The activity is then found from

$\text{Activity} = \lambda N = \left(1.355\times10^{-11}s^{-1}\right)\left(2.666\times10^{21}\right) = 3.612\times10^{10}$ disintegrations/s

The definition of the curie is 1 Ci $= 3.700\times10^{10}$ disintegrations/s. This is approximately equal to the value found above.

8. Over what distance in free space will the intensity of a 5 eV neutron beam be reduced by a factor of one-half? ($T_{1/2} = 12.8$ min).

**Soln.** The speed of the neutron in the beams is found from $\frac{1}{2}mv^2 = K$

$$\frac{1}{2}\left(1.67\times10^{-27}\ kg\right)v^2 = (5\ eV)\left(\frac{1.6\times10^{-19}\ J}{1\ eV}\right)$$

During a time of $T_{1/2} = 12.8$ min, half the neutron will have decayed from the beam. The distance traveled by the undecayed neutrons during this time is

$$d = vt = (31.0\ km/s)(12.8\ \text{min})(60\ s/\text{min}) = 23{,}800\ km$$

or about 2 earth diameters.

9. How much time is required for 5 mg of ${}^{22}Na\left(T_{1/2} = 2.60\ \text{y}\right)$ to reduce to 1 mg?

**Soln.** Since the mass of a sample will be proportional to the number of atoms in the sample, we may write

$$m = m_0e^{-\lambda t} = m_0e^{-\left(0.693/T_{1/2}\right)t}$$

$$\Rightarrow\quad 1\text{ mg} = (5\text{ mg})e^{-(0.693/2.60\text{ y})t}$$

$$\Rightarrow\quad e^{(0.693/2.60\text{ y})t} = 5$$

Taking logarithms of both sides, we find

$$\frac{0.693\,t}{2.60\,y} = \ln 5 = 1.61 \qquad \text{or} \qquad t = 6.04\,y$$

10. If 3.6 gm of uranium is to be completely converted to energy, how many Joules of energy will be obtained from it ? **[H.C.U.-2010]**

(a) $100\times10^{13}$ J (b) $3.24\times10^{13}$ J (c) $32.4\times10^{13}$ J (d) $324.0\times10^{13}$ J

**Soln.** Mass of Uranium $= 3.6\ gm = 3.6\times10^{-3}\,kg$

If the mass of Uranium is to be completely converted to energy, then

$$E = mc^2$$

$$= \left(3.6\times10^{-3}\right)\times\left(3\times10^{8}\right)^2 J = 32.4\times10^{13} J$$

**Correct option is (c)**

11. The half life of thorium is $1.4\times10^{10}$ years. What is the time required for 10% of a sample of thorium to disintegrate ? **[H.C.U.-2011]**

(a) $2.13\times10^{9}$ years (b) $2.13\times10^{7}$ years (c) $2.13\times10^{6}$ years (d) $2.13\times10^{5}$ years

**Soln.** Given : Half life of Thorium $t_{1/2} = 1.4\times10^{10}$ years.

Let the initial number of nuclei of Thorium be $N_0$

$$\therefore\ N = N_0 e^{-\lambda t}$$

Now, $\lambda = \dfrac{\ln 2}{t_{1/2}}$

$$\Rightarrow 0.9N_0 = N_0 e^{-\frac{\ln 2}{t_{1/2}}t}$$

$$\Rightarrow\ 0.9 = e^{-\ln 2(t/t_{1/2})}\ \Rightarrow\ \ln 0.9 = -\ln 2\left(\frac{t}{t_{1/2}}\right)$$

$$\Rightarrow\ t = -t_{1/2}\frac{\ln 0.9}{\ln 2} = t_{1/2}\frac{\ln(10/9)}{\ln 2} = 1.4\times10^{10}\log_2(0.9)$$

$$= 1.4\times10^{10}\times0.152 = 2.13\times10^{9}\ \text{years.}$$

**Correct option is (a)**

12. Let $m_{He}$, $m_p$, $m_n$; $m_e$ represent the masses of $^4$He nucleus, proton, neutron and electron, respectively. These masses satisfy the following relation

(a) $m_{He} = 2\left(m_p + m_n + m_e\right)$ (b) $m_{He} = 2\left(m_p + m_n\right)$

(c) $m_{He} > 2\left(m_p + m_n\right)$ (d) $m_{He} < 2\left(m_p + m_n\right)$ **[H.C.U.-2013]**

**Soln.** For a given nuclei, its mass is less than total mass of its individual constituents (separated protons and neutrons).

$$\therefore\ m_{He} < 2m_p + 2m_n\ \Rightarrow\ m_{He} < 2\left(m_p + m_n\right)$$

**Correct option is (d)**

13. An experiment shows that a given radioactive element emits 100 α particles/minute initially and after 10 seconds the rate becomes 10 α particles/minute. The half life period of this α-emitter (upto 2nd decimal place) is

(a) 3.00 sec. (b) 0.33 sec. (c) 30.00 sec. (d) 0.03 sec. **[H.C.U.-2013]**

**Soln.** Initial activity $\lambda N_i = 100$, final activity $\lambda N_f = 10$

$$\Rightarrow \frac{N_i}{N_f} = 10$$

Now, $N_f = N_i e^{-\lambda t}$, $t = 10$ sec (given)

$$\Rightarrow \frac{1}{10} = e^{-\lambda\times 10} \Rightarrow \lambda = \frac{\ln 10}{10}$$

Half life $t_{1/2} = \frac{\ln 2}{\lambda} \Rightarrow t_{1/2} = \frac{\ln 2}{\ln 10}\times 10 \approx 3$ sec

**Correct option is (a)**

14. The half life of $^{215}$At is 100μs. If the sample contains 6 mg of element, the initial activity of the sample is (Assume Avogadro number is $6.03\times 10^{23}$ atom/mol)

(a) $0.58\times 10^{23}$ Becquerels
(b) $1.16\times 10^{23}$ Becquerels
(c) $3.32\times 10^{23}$ Becquerels
(d) $6.64\times 10^{23}$ Becquerels **[H.C.U.-2014]**

**Soln.** Let initial number of nuclei of A be $N_0$ and that of B be $N_0$

Half life of $A\left(t_{1/2}\right)_A : 3 = \frac{\ln 2}{\lambda_A} \Rightarrow \lambda_A = \frac{\ln 2}{3}$

Half life of $B\left(t_{1/2}\right)_B : 6 = \frac{\ln 2}{\lambda_B} \Rightarrow \lambda_B = \frac{\ln 2}{6}$

After $t = 18$ hours,

$$N_A = N_0 e^{-\lambda t} = N_0 e^{-\frac{\ln 2}{3}\times 18} = N_0 e^{-6\ln 2}$$

$$N_B = N_0 e^{-\lambda_B t} = N_0 e^{-\frac{\ln 2}{6}\times 18} = N_0 e^{-3\ln 2}$$

$$\frac{N_A}{N_B} = \frac{e^{-\ln 2^6}}{e^{-\ln 2^3}} = \frac{1}{2^6} \div \frac{1}{2^3} = \frac{1}{8}$$

$\therefore \; N_A : N_B = 1:8$

**Correct option is (c)**

15. Four moles of a radioactive material decays into 0.5 moles in 90 seconds. Then the disintegration constant is

(a) $\frac{0.693}{30} s^{-1}$ (b) $\frac{0.693}{45} s^{-1}$ (c) $\frac{0.693}{15} s^{-1}$ (d) $\frac{0.693}{90} s^{-1}$ **[H.C.U.-2016]**

**Soln.** Initial number of nuclei = 4 moles $\times N_A$ where $N_A$ is Avogadro number.

Final number of nuclei = 0.5 mole $\times N_A$

We have, $N = N_0 e^{-\lambda t}$

$$\Rightarrow 0.5\, N_A = 4 N_A e^{-\lambda\times 90}$$

$$\Rightarrow \frac{40}{5} = e^{90\lambda} \Rightarrow 90\lambda = \ln 8$$

$$\Rightarrow \lambda = \frac{\ln 8}{90} = \frac{3\ln 2}{90} = \frac{0.693}{30}$$

**Correct option is (a)**

6. Naturally occurring gallium contains two isotopes ${}^{71}_{31}\text{Ga}$ and ${}^{69}_{31}\text{Ga}$ with relative abundance of 40% and 60% respectively. The average number of neutrons in a mixture of gallium nucleus is

(a) 39.2 (b) 37.8 (c) 38.8 (d) 40 **[B.H.U-2010]**

**Soln.** No. of neutrons in ${}^{71}_{31}Ga \Rightarrow N_1 = 40$

No. of neutrons in ${}^{69}_{31}Ga \Rightarrow N_2 = 38$

Relative abundance of ${}^{71}_{31}Ga$ is 40%, relative abundance of ${}^{69}_{31}Ga$ is 60%

The average number of neutrons in a mixture of Gallium nucleus,

$$= N_1 \times \frac{40}{100} + N_2 \frac{60}{100} = 40 \times 0.4 + 38 \times 0.6 = 38.8$$

**Correct option is (c)**

17. If $\lambda_1$ and $\lambda_2$ are the decay constants for parent and daughter nuclei respectively and $N_1(0)$ is the number of parent nuclei at time $t = 0$, then the number of atoms $N_2$ of daughter nuclei at time $t$ is given by **[B.H.U-2011]**

(a) $N_2 = N_1(0)\frac{\lambda_1}{(\lambda_1 - \lambda_2)}\left[e^{-\lambda_2 t} - e^{-\lambda_1 t}\right]$ (b) $N_2 = N_1(0)\frac{\lambda_1}{(\lambda_1 - \lambda_2)}\left[e^{-\lambda_1 t} - e^{-\lambda_2 t}\right]$

(c) $N_2 = N_1(0)\frac{\lambda_2}{(\lambda_2 - \lambda_1)}\left[e^{-\lambda_2 t} - e^{-\lambda_1 t}\right]$ (d) $N_2 = N_1(0)\frac{\lambda_2}{(\lambda_1 - \lambda_2)}\left[e^{-\lambda_1 t} - e^{-\lambda_2 t}\right]$

**Soln.** The number of parent nuclei at time $(t = 0) = N_1(0)$

decay constant for parent nuclei is $\lambda_1$, decay constant for daughter nuclei is $\lambda_2$

$$\frac{dN_1}{dt} = -\lambda_1 N_1 \quad \ldots(i)$$

$$\frac{dN_2}{dt} = +\lambda_1 N_1 - \lambda_2 N_2 \quad \ldots(ii)$$

Solving equation (i)

We have, $N_1 = N_1(0)e^{-\lambda_1 t}$ ...(iii)

Using equation (ii) and (iii),

$$\frac{dN_1}{dt} = \lambda_1 N_1(0)e^{-\lambda_1 t} - \lambda_2 N_2$$

$$\frac{dN_2}{dt} + \lambda_2 N_2 = \lambda_1 N_1(0)e^{-\lambda_1 t}$$

The integrating factor I.F $= e^{\int \lambda_2\, dt} = e^{\lambda_2 t}$

$$\Rightarrow e^{\lambda_2 t}\left(\frac{dN_2}{dt} + \lambda_2 N_2\right) = \lambda_1 N_1(0)e^{(\lambda_2 - \lambda_1)t}$$

$$\Rightarrow \frac{d}{dt}\left(N_2 e^{\lambda_2 t}\right) = \lambda_1 N_1(0)e^{(\lambda_2 - \lambda_1)t}$$

$$\Rightarrow N_2 e^{\lambda_2 t} = \frac{\lambda_1}{\lambda_2 - \lambda_1} N_1(0) e^{(\lambda_2 - \lambda_1)t} + C$$

Where $C$ is constant.

At $t = 0, N_2(t = 0) = N_2(0) = 0$

$$C = N_2(0) - \frac{\lambda_1}{\lambda_2 - \lambda_1} N_1(0) \Rightarrow C = -\frac{\lambda_1}{\lambda_2 - \lambda_1} N_1(0)$$

$$\Rightarrow N_2 e^{\lambda_2 t} = \frac{\lambda_1}{\lambda_2 - \lambda_1} N_1(0) e^{(\lambda_2 - \lambda_1)t} - \frac{\lambda_1}{\lambda_2 - \lambda_1} N_1(0)$$

$$\Rightarrow N_2 e^{\lambda_2 t} = \frac{\lambda_1}{\lambda_2 - \lambda_1} N_1(0)\left[e^{\lambda_2 t - \lambda_1 t} - 1\right]$$

$$\Rightarrow N_2(t) = \frac{\lambda_1}{\lambda_2 - \lambda_1} N_1(0)\left[e^{-\lambda_1 t} - e^{-\lambda_2 t}\right]$$

$$\Rightarrow N_2(t) = \frac{\lambda_1}{\lambda_1 - \lambda_2} N_1(0)\left[e^{-\lambda_2 t} - e^{-\lambda_1 t}\right]$$

$\therefore$ **Correct option is (a)**

18. Cobalt 60 is used in hospitals as a radiaoctive source in medicines. It has a half-life of 5.25 years. After a use for certain period its activity has been found to decrease by a factor of 8. How is old the sample?

(a) 15.75 years (b) 10.5 years (c) 21 years (d) 42 years **[B.H.U-2011]**

**Soln.** Half life of Cobalt-60, $t_{1/2} = 5.25$ years

decay constant $\lambda = \frac{\ln 2}{t_{1/2}} = \frac{\ln 2}{5.25}$

Now, initial activity $A_1 = \lambda N_1$ ...(i)

Where $N_1$ – initial no. of nuclei

Final acitvity, $A_2 = \lambda N_2$ ...(ii)

Given, $A_2 = \frac{A_1}{8}$ ...(iii)

$$\Rightarrow \lambda N_2 = \frac{\lambda N_1}{8}$$

$$\Rightarrow N_2 = \frac{N_1}{8}$$

We have, $N_2 = N_1 e^{-\lambda t}$

$$\Rightarrow \ln 8 = \lambda t = \frac{t \ln 2}{5.25}$$

$$t = 5.25 \frac{\ln 8}{\ln 2} = 5.25 \log_2 8 = 5.25 \log_2 2^3 = 5.25 \times 3 = 15.75 \text{ years}$$

**Correct option is (a)**

9. The interaction between neutrons and protons may be represented by the Yukawa potential $u(r) = -\left(\frac{r_0}{r}\right)u_0 e^{-r/a_0}$, then the force $F(r)$ between neutrons and protons is **[B.H.U-2011]**

(a) $-\left(\frac{r_0}{r}\right)u_0 e^{-r/r_0}\left[\frac{1}{r}-\frac{1}{r_0}\right]\hat{r}$ (b) $-\left(\frac{r_0}{r}\right)u_0 e^{-r/r_0}\left[\frac{1}{r}+\frac{1}{r_0}\right]\hat{r}$

(c) $-\left(\frac{r_0}{r}\right)u_0 e^{-r/r_0}\left[\frac{1}{r_0}\right]\hat{r}$ (d) $-\frac{r_0}{r^2}u_0 e^{-r/r_0}\left[\frac{1}{r}-\frac{2}{r_0}\right]$

**Soln.** Given $U(r) = -\left(\frac{r_0}{r}\right)u_0 e^{-r/r_0}$

$$F(r) = \frac{-\partial U(r)}{\partial r}$$

$$= -\frac{\partial}{\partial r}\left[-\frac{r_0}{r}u_0 e^{-r/r_0}\right]$$

$$= r_0 u_0 \frac{\partial}{\partial r}\left[\frac{e^{-r/r_0}}{r}\right]$$

$$= r_0 u_0\left[-\frac{e^{-r/r_0}}{r_2} - \frac{1}{rr_0}e^{-r/r_0}\right]$$

$$F(r) = -\frac{r_0}{r}u_0 e^{-r/r_0}\left[\frac{1}{r}+\frac{1}{r_0}\right]\hat{r}$$

**Correct option is (b)**

20. A radioactive nucleus of type 1 decays exponentially with a decay constant $\lambda_1$ to stable nucleus of type 2. If at time $t = 0$, the numbers of type 1 and 2 nuclei are respectively $N_1 = N_0$ and $N_2 = 0$, what is the number of type 2 nucleus present at time $t$? **[B.H.U-2012]**

(a) $N_0 \exp - \lambda_1 t$ (b) $N_0(1-\exp - \lambda_1 t)$ (c) $N_0(1+\exp - \lambda_1 t)$ (d) $1 - N_0 \exp - \lambda_1 t$

**Soln.** For nucleus of type 1,

$$\frac{dN_1}{dt} = -\lambda_1 N_1 \qquad \ldots\text{(i)}$$

For nucleus of type 2,

$$\frac{dN_2}{dt} = \lambda_1 N_1 \qquad \ldots\text{(ii)}$$

Now, from (i),

$$\frac{1}{N_1}dN_1 = -\lambda_1 dt$$

Integrating both sides,

$$\ln N_1 = -\lambda_1 t + c$$

at $t = 0, N_1 = N_0, c = \ln N_0$

$\Rightarrow \ln N_1 = -\lambda_1 t + \ln N_0$

$\Rightarrow N_1 = N_0 e^{-\lambda_1 t}$

From (ii), $\frac{dN_2}{dt} = \lambda_1 N_1 = \lambda_1 N_0 e^{-\lambda_1 t}$

$\Rightarrow dN_2 = \lambda_1 N_0 e^{-\lambda_1 t} dt$

$\Rightarrow N_2 = \lambda_1 N_0 \frac{e^{-\lambda_1 t}}{-\lambda_1} + D$

Now, at $t = 0, N_2 = 0 \Rightarrow D = N_0$

$\Rightarrow N_2 = -N_0 e^{-\lambda_1 t} + N_0$

$\Rightarrow N_2 = N_0 \left(1 - e^{-\lambda_1 t}\right)$

**Correct option is (b)**

21. Suppose that ${}^{A}_{Z}X$ decays by natural radioactivity in two stages to ${}^{A-4}_{Z-1}Y$. The two stages would most likely be which of the following? **[B.H.U-2012]**

| | First Stage | Second Stage |
|---|---|---|
| (a) | $\beta^-$ emission with an antineutrino | $\alpha$ emission |
| (b) | $\beta^-$ emission | $\alpha$ emission with a neutrino |
| (c) | $\beta^-$ emission | $\gamma$ emission |
| (d) | Emission of a deuteron | emission of two neutrons |

**oln.** First state :

$${}^{A}_{Z}X \longrightarrow {}^{A}_{Z+1}X + {}^{0}_{-1}e + {}^{0}_{0}\bar{v} \qquad \text{(beta decay with an anti-nentrino)}$$

Second state:

$${}^{A}_{Z+1}X \longrightarrow {}^{A-4}_{Z-1}X + {}^{4}_{2}He \qquad \text{(alpha decay)}$$

**Correct option is (a)**

2. ${}^{238}_{92}$U nucleus decay involves 8 $\alpha$- decays and 6 $\beta$- decays. The end product of the series have:
(a) Z = 82, A = 206 (b) Z = 84, A = 224 (c) Z = 88, A = 206 (d) Z = 84, A = 212 **[B.H.U-2014]**

**oln.** For uranium $A = 238, Z = 92$

In an alpha decay mass number A reduces by 4 and atomic number $Z$ reduced by 2.

In a beta decay $A$ remains unchanged and $Z$ increases by 1.

Therefore, the new mass number $= 238 - 8 \times 4 \Rightarrow A = 206$

The new atomic number $= 92 - 8 \times 2 + 6 \times 1 \Rightarrow Z = 82$

**Correct options is (a)**

23. What is the number of $\alpha$ decays that occur in a 1 gm sample of thorium in 1 year. The disintegration constant of thorium (232) is $1.58\times10^{-18}\ \text{sec}^{-1}$ and Avogadro's number is $6.023\times10^{23}$? **[B.H.U-2011]**

(a) $13\times10^{12}$ (b) $13\times10^{10}$ (c) $6\times10^{11}$ (d) $11\times10^{15}$

**Soln.** Let the number of α decays be $x$ per second. 1$g$ of Thorium–232 contains = $\dfrac{6.023\times10^{23}}{232}$ atoms of Th–232

Decay constant or disintegration constant

$$\lambda=\frac{x \text{ per sec}}{\dfrac{6.023\times10^{23}}{232}}$$

$$\therefore \text{ Alpha decays per sec, } x=\frac{\lambda\times6.023\times10^{23}}{232}$$

$$x=\frac{1.58\times10^{-18}\times6.023\times10^{23}}{232}$$

$$=0.04\times10^{5} \text{ particles sec}^{-1}$$

For 1 year, the no. of alpha decays

$= (1 \text{ year}) \times 0.04 \times 10^5\ \text{s}^{-1}$

$= (3600 \times 24 \times 365) \times 0.04 \times 10^5 \approx 13 \times 10^{10}$

**Correct option is (b)**

24. Consider the fusion reaction $6\,{}_1^2H \to 2\,{}_2^4He+2p+2n+Q(43\ MeV)$. The total energy detained by fusing 1 kg of deuterium to form helium is **[B.H.U-2011]**

(a) $4.3\times10^{27}\ MeV$ (b) $8.6\times10^{27}\ MeV$ (c) $5.2\times10^{27}\ MeV$ (d) $2.15\times10^{27}\ MeV$

**Soln.** No. of deuterium atoms in 1 kg of deuterium $=\dfrac{N_A}{2}\times1000$

where $N_A=6.023\times10^{23}$

Now, each reaction requires 6 deuterium atoms.

$$\text{Number of reactions possible} =\frac{1}{6}\times\left(\frac{6.023\times10^{23}\times10^{3}}{2}\right)$$

$Q$-value for each reaction = 43 MeV

Total energy detained by fusing 1 kg of deutorium = (Number of reactions) × $Q$-value of one reaction.

$$=\frac{1}{6}\times\frac{6.023\times10^{23}\times10^{3}}{2}\times43\text{ MeV}$$

$= 2.15 \times 10^{27}$ MeV

**Correct option is (d)**

25. What is the amount of energy released when 10 micrograms of uranium ($^{235}$U) undergoes fission, if energy released per fission is 200 MeV? **[B.H.U-2012]**

(a) $5.126 \times 10^{18}$ MeV (b) 5.126 MeV (c) 5.125 MeV (d) $5.126 \times 10^{6}$ MeV

**Soln.** Number of Uranium–235 atoms in 10 $\mu g$ of Uranium–235

$$= \frac{6.023 \times 10^{23}}{235} \times 10 \times 10^{-6}$$

Total energy released = Number of $U^{235}$ atoms × energy per fission

$$= \frac{6.023 \times 10^{18}}{235} \times 200 \text{ MeV} \approx 5.126 \times 10^{18} \text{ MeV}$$

**Correct option is (a)**

26. A pure Fermi-transition in the nuclear β- decay is the one where: **[B.H.U-2016]**

(a) $\Delta J = \pm 1, O^+ \to O^+$ (b) $\Delta J = 0, O^+ \to O^+$

(c) $\Delta J = \pm 2, O^+ \to O^+$ (d) $\Delta J > 0, O^+ \to O^+$

**Soln.** For pure Fermi-transition in the nuclear β-decay, $\Delta J = 0$ and parity does not change.

Only one among the options, i.e., option (b), satisfies the above criteria for pure Fermi-transition in β-decay.

$\Delta J = 0$, and parity remains same $0^+ \to 0^+$

27. The following nuclear β- decay $\underset{0^+}{\overset{6}{\text{He}}} \to \underset{1^+}{\overset{6}{\text{Li}}} + e^- + \bar{\nu}, |\Delta J| = 1$ is an explict example of the **[B.H.U-2016]**

(a) Forbidden transition (b) Mixed transition

(c) Pure Fermi transition (d) Pure Gamow Teller transition

**Soln.** $\underset{0^+}{\overset{6}{He}} \to \underset{1^+}{\overset{6}{Li}} + e^- + \bar{\nu}$

Given: $|\Delta J| = 1$

For allowed nuclear β-decay, $\Delta J = 1$ and partiy remains the same $0^+ \to 1^+$. So this is a pure Gamow Teller transition.

28. Geiger's law for the "Range Energy" relationship for the α- particle is well known infact, the range R and velocity v of the α- particle are related by **[B.H.U-2016]**

(a) $R \,\alpha\, v^5$ (b) $R \,\alpha\, v^4$ (c) $R \,\alpha\, v^3$ (d) $R \,\alpha\, v^2$

**Soln.** Geiger's law for the "Range Energy" relationship for the α-particle is

$$R = bE^{3/2} \quad \text{...(i)}$$

Where $b$ is a constant.

Now, $E = \frac{1}{2}mv^2 \Rightarrow E \propto v^2$ ...(ii)

Where $v$ is velocity and $m$ is mass of particle.

Using (ii) in (i),

$$R \propto \left(v^2\right)^{3/2}$$

Therefore, $R \propto v^3$

**Correct option is (c)**

29. A sample of radioactive nuclie of a certain element can decay only by γ-emission and β-emission. If the half-life for γ-emission is 24 minutes and that for β-emission is 36 minutes, the half-life for the sample is

(a) 30 minutes (b) 24 minutes (c) 20.8 minutes (d) 14.4 minutes **[B.H.U-2012]**

**Soln.** Given: Half life of γ-decay : $\left(t_{1/2}\right)_\gamma = 24$ minutes

Half life of β-emission : $\left(t_{1/2}\right)_\beta = 36$ minutes

Decay constant for γ-decay : $\lambda_\gamma = \dfrac{\ln 2}{\left(t_{1/2}\right)_\gamma} = \dfrac{\ln 2}{24}$

Decay constant for β-emission : $\lambda_\beta = \dfrac{\ln 2}{\left(t_{1/2}\right)_\beta} = \dfrac{\ln 2}{36}$

Let the decay constant and half life of sample be $\lambda_s$ and $\left(t_{1/2}\right)_s$ respectively.

$$\lambda_s = \lambda_\gamma + \lambda_\beta$$

$$\Rightarrow \frac{\ln 2}{\left(t_{1/2}\right)_s} = \frac{\ln 2}{\left(t_{1/2}\right)_\alpha} + \frac{\ln 2}{\left(t_{1/2}\right)_\beta} \Rightarrow \frac{1}{\left(t_{1/2}\right)_s} = \frac{1}{24} + \frac{1}{36}$$

$$\Rightarrow \left(t_{1/2}\right)_s = \frac{24\times 36}{24+36} = 14.4 \text{ minutes}$$

**Correct option is (d).**

30. An 8-centimetre diameter by 8-centimetre long NaI(Tl) detector detects gamma rays of a specific energy from a point source of radioactivity. When the source is placed just next to the detector at the center of the circular face, 50 percent of all emitted gamma rays at the energy are detected. If the detector is moved to 1 meter away, the fraction of detected gamma rays drops to **[B.H.U-2012]**

(a) $10^{-4}$ (b) $2\times 10^{-4}$ (c) $4\times 10^{-4}$ (d) $8\pi\times 10^{-4}$

**Soln.** For a point source, fraction of gamma rays falling on detector $\alpha = \dfrac{\pi r^2}{4\pi l^2}$, where $r$ is radius of detector, $l$ is the distance between source and detector.

Given: $r = 4$ cm, $l = 100$ cm

$$\alpha = \frac{\pi\times 4^2}{4\pi\times 100^2} = 4\times 10^{-4}$$

Also, the detector detects only 50% of the gamma rays falling on it.

The fraction of detected gamma rays $= \dfrac{\alpha}{2} = 2\times 10^{-4}$

**Correct option is (b)**

31. Geiger-Nuttal law gives the relationship between **[B.H.U-2011]**

(a) energy and range of $\alpha$ particle
(b) half-life and decay constant of $\alpha$ particle
(c) half-life and energy of $\alpha$ particle
(d) half-life of $\alpha$ emitter and energy of $\alpha$ particle

**Soln.** Geiger-Nuttal law gives the relationship between the decay constant of a radio isotope with the energy of the alpha particles. In other words, the law relates the half life of α emitter and energy of α particle.

**Correct option is (d).**

32. The half-life of a radio isotope is 5 years. The fraction of atoms decayed in this isotope in a period of 20 years will be **[B.H.U-2011]**

(a) $\frac{1}{16}$ (b) $\frac{7}{8}$ (c) $\frac{1}{8}$ (d) $\frac{15}{16}$

**Soln.** Given: $t_{1/2} = 5$ years

$$\therefore\ \lambda = \frac{\ln 2}{t_{1/2}} = \frac{\ln 2}{5}$$

We know, $N = N_0 e^{-\lambda t}$

$$\Rightarrow\ N = N_0 e^{-\frac{\ln 2}{5}\times 20} = N_0 e^{-4\ln 2}$$

$$\Rightarrow\ N = \frac{N_0}{16}\ \text{(Fraction of atoms remaining)}$$

$\therefore$ Fraction of atoms decayed $= N_0 - \frac{N_0}{16} = \frac{15}{16}N_0$

**Correct option is (d)**

33. A reactor is developing energy at the rate of $32 \times 10^6$ watts. How many atoms of $^{235}$U undergo fission per second ? **[B.H.U-2012]**

(Assume that on the average, an average of 200 MeV is released per fission).

(a) $10^6$ (b) $10^{12}$ (c) $10^{18}$ (d) $10^{24}$

**Soln.** Energy released per second = $32 \times 10^6$ J

Energy released per fission = 200 MeV

$= 200 \times 10^6 \times 1.6 \times 10^{-19}$ J

$= 32 \times 10^{-12}$ J

Number of atoms of $^{235}$U undergo fission per sec $= \frac{32\times10^6}{32\times10^{-12}} = 10^{18}$ atoms/second

**Correct option is (c)**

34. The binding energy of the neon isotope $^{20}_{10}$Ne is 160.65 MeV. Thus, its atomic mass, in atomic unit ($u$), would be **[B.H.U-2015]**

(a) 19.99 $u$ (b) 18.99 $u$. (c) 17.99 $u$ (d) 16.99 $u$

**Soln.** Binding energy $B = \Delta m\, c^2$, where $\Delta m$ is mass defect and $c$ is speed of light.

$$\therefore\ \Delta m = \frac{B}{c^2} = \frac{160.65\ \text{MeV}}{\left(3\times10^8\ \text{m/s}\right)^2}$$

$$= \frac{160.65\times10^6\times1.6\times10^{-19}}{\left(3\times10^8\right)^2} = 28.56\times10^{-29}\,g$$

We know 1 gram = $6.022 \times 10^{23}$ a.m.u.

$\Delta m = 28.25 \times 10^{-29} \times 6.022 \times 10^{23} = 170.1215 \times 10^{-16} \approx 0.00017$

Atomic mass $= A - \Delta m = 20 - 0.00017 \approx 19.99\,u$

**Correct option is (a)**

35. If a 2 MeV alpha particle $\left({}_2He^4\right)$ incident on ${}_{47}Ag^{107}$ nucleus is scattered at 90°, then the impact parameter is $\left(e^2/4\pi\epsilon_0 = 1.44 \text{ MeV fm}\right)$

(a) $3.38\times10^{-15}$ m (b) $47\times10^{-5}$ m (c) $6.78\times10^{-15}$ m (d) 0 m **[H.C.U.-2012]**

**Soln.** K.E. of alpha particle $\left({}_2He^4\right) = 2\text{ MeV} = 2\times10^6\times1.6\times10^{-19}J = 3.2\times10^{-13}J$

Scattering angle $\theta = 90°$

$Z_1 = 2,\ Z_2 = 47$

Impact parameter $b = \dfrac{Z_1Z_2ke^2}{2(K.E.)}\cot\dfrac{\theta}{2}$, where $k = \dfrac{1}{4\pi\epsilon_0} = 9\times10^9 N$

$$b = \frac{2\times47\times9\times10^9\times\left(1.6\times10^{-19}\right)^2}{2\times3.2\times10^{-13}}$$

$$= \frac{47\times9\times10^9\times10^{-38}\times1.6\times1.6}{2\times1.6\times10^{-13}} = 7.2\times47\times10^{-16}$$

$$= 338.4\times10^{16} = 33.8\times10^{-15}\,m$$

**Note:** option (a) is misprinted in original question paper.

**Correct option is (a)**

36. It two deuterons are at a distance of $10^{-12}$ cm, the nuclear force between them just overcomes the repulsive electrostatic force. The height of the electrostatic potential barrier is **[B.H.U-2010]**

(a) zero (b) 0.14 MeV (c) 0.31 MeV (d) 0.49 MeV

**Soln.** The electrostatic Potential barrier is

given by $V = \dfrac{e^2}{4\pi\epsilon_0}\dfrac{Z_a+Z_b}{R_a+R_b}$

For two deuterons, $Z_a = 1, Z_b = 1$

$R_a + R_b = 10^{-12}$ cm $= 10^{-14}$ m

We know, $\dfrac{e^2}{4\pi\epsilon_0} = 1.44\times MeV\ fm$

$= 1.44\times10^{-15}\ MeV\ m$

$$V = \frac{1.44\times10^{-5}\times1\times1}{10^{-14}} = 0.144\ MeV \approx 0.14\ MeV$$

**Correct option is (b)**

37. The reaction $\pi^- p \to \Lambda^\circ\pi^\circ$ in strong interaction is not possible because **[B.H.U-2010]**

(a) it does not conserve charge (b) it does not conserve baryon number

(c) it does not conserve strangeness (d) it does not conserve isospin

**Soln.** The reaction is :

$\pi^- + p \to \Lambda^\circ + \pi^\circ$

Charge $Q: -1+1 \to 0+0 \Rightarrow \Delta Q = 0$ $Q \to$ conserved

Baryon No. $B:0+1\rightarrow+1+0\Rightarrow\Delta B=0$ $B\rightarrow$ conserved

Strangeness $S:0+1\rightarrow-1+0\Rightarrow\Delta S\neq0$ $S\rightarrow$ not conserved

Isospin $I:1+\frac{1}{2}\rightarrow0+1\Rightarrow\Delta I\neq0$ $I\rightarrow$ not conserved

It is a strong reaction where the charge and the Baryon number are conserved. However, since the strangeness number and isospin are not conserved, the reaction is not possible.

**Correct options are (c, d)**

38. The ground state of a deuteron is **[B.H.U-2016]**

(a) A pure $^3S_1$ state (b) A pure $^3P_1$ state

(c) A mixture of $^3S_1$ and $^3P_1$ states (d) A mixture of $^3S_1$ and $^3D_1$ states

**Soln.** The reason for mixed states of deuteron wave functions arises from the fact that $\mu_d-(\mu_p+\mu_n)\neq0$, where $\mu_d$ is the experimental magnetic dipole, $\mu_p$ and $\mu_n$ are the magnetic dipole moments of proton and neutron, respectively.

The ground state of the deuteron has total angular momentum $J=1$.

The corresponding states : $^1P_1, {}^3P_1, {}^3S_1$ and $^3D_1$. For same parity, one can have two pairs $^1P_1$ and $^3P_1$; $^3S_1$ and $^3D_1$.

Only $^3S_1$ and $^3D_1$ gives proper result with $^3S_1$ state = 96% and $^3D_1$ state = 4%

**Correct option is (d)**

39. According to the nuclear shell-model, the spin and parity of the ground state of the nucleus $_{19}K^{39}$ would be

(a) $\left(\frac{3}{2}\right)^+$ (b) $\left(\frac{1}{2}\right)^+$ (c) $\left(\frac{3}{2}\right)^-$ (d) $\left(\frac{1}{2}\right)^-$ **[B.H.U-2016]**

**Soln.** Given : $_{19}K^{39}\Rightarrow Z=19, A=39$

No. of neutrons $N=39-19=20$

$Z=19\,(odd)\rightarrow\left(1S_{1/2}^2\right)\left(1P_{3/2}^4\right)\left(1P_{1/2}^2\right)\left(1d_{5/2}^6\right)\left(2s_{1/2}^2\right)\left(1d_{3/2}^3\right)$

For last unpaired nucleon,

$j=\frac{3}{2}$, parity $=(-1)^l=(-1)^2$ $(\because l=2)$

Therefore, spin parity $=\left(\frac{3}{2}\right)^+$

**Correct options are (a)**

40. The quark content of a neutron is:

(a) uud (b) udd (c) uds (d) $ud\bar{s}$ **[B.H.U-2016]**

**Soln.** The quark content of a neutron is 'udd'

$$\text{Since it has charge } Q=\begin{pmatrix} \overset{u}{\frac{2}{3}} & \overset{d}{-\frac{1}{3}} & \overset{d}{-\frac{1}{3}} \end{pmatrix}=0$$

and it has Baryon number $B=\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)=1$

It has strangeness number $S=(0+0+0)=0$

$I_3=\frac{1}{2}-\frac{1}{2}-\frac{1}{2}=-\frac{1}{2}$

**Correct option is (b)**

1. Given that $^{113}$Cd has density $n=5.58\times10^{27}$ atoms/m$^3$ and its capture cross-section $\sigma=2\times10^{-24}\text{ m}^2$ for thermal neutrons, the mean-free path of these neutrons in $^{113}$Cd is **[B.H.U-2015]**

(a) $8.93\times10^{-7}$ m (b) $8.93\times10^{-5}$ m (c) $8.93\times10^{-9}$ m (d) $8.93\times10^{-10}$ m

**oln.** The mean free path

$\lambda=\frac{1}{\Sigma}$ (the reciprocal of the macroscopic cross-section $\Sigma$)

or, $\lambda=\frac{1}{n\sigma}$

When $n$ is atomic density and $\sigma$ is capture cross-section.

$n=5.58\times10^{27}$ atoms/m$^3$, $\sigma=2\times10^{-24}\text{ m}^2$

$$\lambda=\frac{1}{5.58\times10^{27}\times2\times10^{-24}\text{ m}}$$

$$\approx\frac{1}{1.12\times10^{4}}\text{ m}\approx8.93\times10^{-5}\text{ m}$$

**Correct option is (b)**

42 The life time of a free neutron to decay into a proton, an electron and an antineutrino is approximately

(a) $10^{-23}$ sec (b) $10^{-20}$ sec (c) $10^{-8}$ sec (d) $10^{3}$ sec **[B.H.U-2016]**

**Soln.** A free neutron decays into a proton, an electron and an anti-neutrino, is an example of weak force decay. The life time is about 15 minutes = 900 seconds ~ $10^3$s

**Correct option is (d)**

## PRACTICE SET

1. Find the ratio of the nuclear to the atomic density for hydrogen. (Assume the nuclear radius to be 1 fm). [$0.15 \times 10^{15}$]

2. A nucleus with $A = 235$ splits into two nuclei whose mass numbers are in the ratio 2:1. Find the radii of the new nuclei. [$r_1 = 5.99$ fm, $r_2 = 7.55$ fm]

3. Calculate the ratio of the nuclear radius of ${}^{208}_{82}Pb$ to the radius of its innermost electrons as calculated from Bohr theory. [1/77.8]

4. What is the mass of a sample of ${}^{14}_{6}C\left(T_{1/2} = 5570\text{ y}\right)$ that has an activity of 5 Ci. [1.09 g]

5. What is the activity of $5 \times 10^{-7}$ kg of ${}^{230}_{92}U$, whose half-life is $0.180 \times 10^{7}$ s? [14.9 Ci]

6. How much time is required for an amount of ${}^{90}_{38}Sr\left(T_{1/2} = 28\text{ y}\right)$ to be reduced by 75%? [56 y]

# Chapter 9

# Special Theory of Relativity

## 9.1: Reference Frame:

A system of coordinates axes which defines the position of particle or an event in two or three dimensional space is called frame of reference.

A reference frame with four coordinates x, y, z, t is referred as space-time frame.

Frame of references (i) Inertial (ii) Non-inertial

1. **Inertial frame of reference:** Newton's law does hold in this frame of reference. This is also called Newtonian, Galilean or inertial frame of reference. All frame of reference, moving with constant velocity with respect to an inertial frame, are also inertial frame.

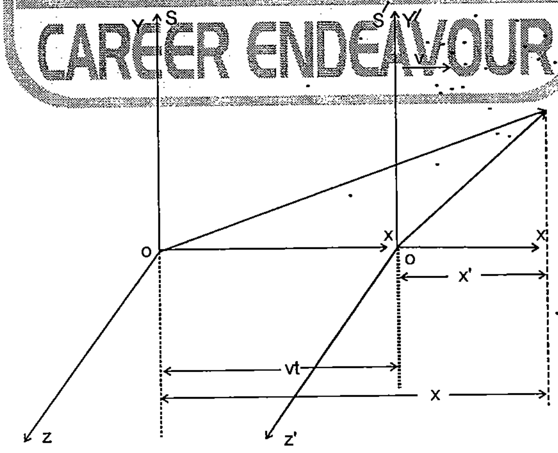
   Inertial frames are non-accelerating frames.

2. **Non-inertial frame:** Accelerated frame of reference in which Newton's law does not hold good.

## 9.2. Galilean transformation:

An event or a physical phenomenon, observed simultaneously in two separate sets of coordinates corresponding to two frame of references. If two systems are inertial, the transformation relating these two different coordinates is called Galilean transformation.

**(a) Transformation of position:**



An event or physical phenomena, observed simultaneously in two separate reference frames, will have two separate set of coordinates. Equation relating two coordinates is called transformation equation. And if two system happen to be inertial or Galilean one, the transformation is reffered as Galilean transformation.

Let S and S' be two inertial frame of reference with observer at origin O and O' moving with a relative velocity v along +ve direction of x.

Now if we assume time to be independent of any frame of reference, we have four transformation equation from S to S'.

$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\}$$

**(b) Transformation of distance or length:** Let two events occurring simultaneously at $A_1$ and $A_2$, some distance apart, along axes of x, such that their coordinates in two frame of references are $x_1, y, z$ ; $x_2, y, z$ and $x_1, y', z'$; $x_2', y', z'$ respectively.

Transformation equation. for event at $A_1$, (from S to S'): $x_1' = x_1 - vt,\ y' = y,\ z' = z$

Transformation equation for event at $A_2$ (from S to S'): $x_2' = x_2 - vt,\ y' = y,\ z' = z$

Hence distance between two events

$$x_2' - x_1' = \Delta x' = (x_2 - vt) - (x_1 - vt) = (x_2 - x_1) = \Delta x$$

Now, when a particular property is characterized by the same number in two frame of reference, it is said to be "invariant to transformation."

Hence, distance or length is invariant to Galilean transformation.

**(c) Transformation of velocity:** Now consider the velocity measurement u and u' as measured by two observer in their frame of reference. Thus, if $x_1, t_1$ and $x_2, t_2$ be co-ordinates of initial & final events in frame S and $x_1', t_1'$; $x_2', t_2'$ be co-ordinates of initial and final events in frame S'.

Velocity relative to frame S, $u = \dfrac{\Delta x}{\Delta t} = \dfrac{x_2 - x_1}{t_2 - t_1}$, since,

$$\begin{aligned} x_1 &= x_1' + vt_1' \\ x_2 &= x_2' + vt_2' \\ t_1 &= t_1' \\ t_2 &= t_2' \end{aligned}$$

$$u = \frac{(x_2' + vt_2') - (x_1' + vt_1')}{t_2' - t_1'} = \frac{x_2' - x_1'}{t_2' - t_1'} + \frac{v(t_2' - t_1')}{(t_2' - t_1')}$$

$$\Rightarrow u = u' + v$$

**(d) Transformation of acceleration:** Let 'a' and 'a'' be the acceleration in S and S' frame of reference.

We have, $a' = \dfrac{du'}{dt} = \dfrac{d(u - v)}{dt} = \dfrac{du}{dt} - 0 = a \quad [\because \text{ v is constant}]$

$\therefore$ Acceleration in invariant under Galilean transformation.

## 9.3: Ether Hypothesis and Michelson Morley experiment:

Ether hypothesis is an analogy between light and sound propagation. According to this hypothesis, electromagnetic waves are carried by a medium called the ether. The reference frame in which the ether is at rest would constitute an absolute coordinate system.

The value of the speed of light would then hold just in the reference frame in which the ether is at rest.

In particular, the existence of an ether would imply that electromagnetic waves in a moving reference system propagate (analogous to sound propagation) with distinct velocities forward (i.e., direction of motion of the source) and sideways. This hypothesis was tested for the ûrst time in the Michelson-Morley experiment (1887) by means of a Michelson interferometer. Here one observes with an interference setup whether the speed of light changes because of Earth's motion. The moving system in which the experiment was performed is Earth itself on its path around the Sun.

The experiment proves that light propagates with equal velocity c along Earth's orbit and in the perpendicular direction, disproving the ether hypothesis.

## 9.4: Postulates of special relativity

(1) **Principle of relativity:** The laws of Physics are same in all inertial frames.

(2) **Principle of constancy of speed of light:** The speed of light in empty space (called vacuum) is same in all inertial frames of reference. The velocity of light is 3 x $10^8$ m/s in free space.

**Invariant Interval:** The quantity : $\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2\Delta t^2$ is called interval. It is invariant in all frames.

## 9.5: Lorentz Transformation:

Let S and S' be two reference frames, with their axes parallel and S' moving relative to S with a uniform velocity $v$ along $x$-axis. Let the origin O and O' of the systems coincide at the instant $t = t' = 0$, when a light originates at O ( and O' ).

Then, by Einstein's postulates, the wavefront of light at time $t$ in S-system would be a sphere given by

$$x^2 + y^2 + z^2 = c^2t^2$$

where c is the velocity of light, and the corresponding equation in $S'$-system will be

$$x'^2 + y'^2 + z'^2 = c^2t'^2$$

The relative motion of the frames does not change the transverse co-ordinates; $y' = y$ and $z' = z$. So, we may write

$$x^2 - c^2t^2 = x'^2 - c^2t'^2 \qquad \text{...(1)}$$

For a complete equivalence of the two inertial frames $S(x, y, z, t)$ and $S'(x', y', z', t')$, the transformation quations must be symmetrical w.r.t. both and if quantities $(x, y, z, t)$ be finite, $(x', y', z', t')$ must also be so. The above conditions require that the transformation equations should be linear. With these considerations, let

$$x' = \gamma(x - vt) \qquad \text{... (2)}$$

$$x = \gamma'(x' + vt') \qquad \text{... (3)}$$

where $\gamma$ and $\gamma'$ are undertermined constants

$\therefore \quad \dfrac{x}{\gamma'} = x' + vt'$, from (3)

$\Rightarrow \quad t' = \dfrac{1}{v}\left(\dfrac{x}{\gamma'} - x'\right) = \dfrac{1}{v}\left[\dfrac{x}{\gamma'} - \gamma(x - vt)\right]$ using (1)

$t' = \gamma\left[t - \dfrac{x}{v}\left(1 - \dfrac{1}{\gamma\gamma'}\right)\right]$, on rearrangement ... (4)

Substituting for $x'$ from (2) and $t'$ from (4) in (1)

$$x^2 - c^2t^2 = \gamma^2(x - vt)^2 - c^2\gamma^2\left[t - \frac{x}{v}\left(1 - \frac{1}{\gamma\gamma'}\right)\right]^2$$

Equating the coefficients of $t^2$ and $x^2$ from both sides.

$$-c^2 = \gamma^2 v^2 - c^2\gamma^2 \qquad \dots (5)$$

And $$1 = \gamma^2 - c^2\gamma^2 \cdot \frac{1}{v^2}\left(1-\frac{1}{\gamma\gamma'}\right)^2 \qquad \dots (6)$$

$\therefore$ $\gamma^2\left(c^2 - v^2\right) = c^2$, from (5)

$$\Rightarrow \quad \gamma = \frac{c}{\sqrt{c^2 - v^2}} = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad \dots (7)$$

And $\dfrac{1}{\gamma^2} = 1 - \dfrac{c^2}{v^2}\left(1 - \dfrac{1}{\gamma\gamma'}\right)^2$, from (6)

$\Rightarrow$ $1 - \dfrac{v^2}{c^2} = 1 - \dfrac{c^2}{v^2}\left(1 - \dfrac{1}{\gamma\gamma'}\right)^2$, using (7)

$$\Rightarrow \quad 1 - \frac{1}{\gamma\gamma'} = \frac{v^2}{c^2} \Rightarrow \frac{1}{\gamma\gamma'} = 1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2}, \text{ using (7)} \qquad \dots (8)$$

$$\therefore \quad \gamma' = \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \qquad \dots (9)$$

So, the required Lorentz transformation equations are

$$\boxed{\begin{aligned} x' &= \frac{(x - vt)}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{\left(t - vx/c^2\right)}{\sqrt{1 - v^2/c^2}} \end{aligned}}$$

The conjugate transformation relations are obtained simply by replacing v and –v in the above set and interchanging the primed and unprimed quantities. So, the transformation relations from $S'$ to $S$-sytem are :

$$\boxed{\begin{aligned} x &= \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \\ y &= y' \\ z &= z' \\ t &= \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}} \end{aligned}}$$

## 9.6: Length contraction:

The length of a rod, as measured in a frame of reference, at rest with respect to observer, is called its proper length and obviously be same in all stationary frame of reference. It is only in moving frame that it will be different. Let a rod laid along the axis of x is frame of reference S, at rest with respect to the observer and let $x_1$ and $x_2$ be the co-ordinates of its two ends.

$\therefore$ Proper length of rod, $L_0 = (x_2 - x_1)$

If x–coordinates of the ends of the rod in reference frame $S'$, moving with uniform velocity v with respect to frame S be $x_1'$ and $x_2'$, as noted simultaneously at same instant $t'$, we have length of rod in moving reference frame $S'$ given by $L = (x_2' - x_1')$.

From inverse Lorentz transformation:

$$x_1 = \frac{x_1' + vt'}{\sqrt{1 - v^2/c^2}}; \quad x_2 = \frac{x_2' + vt'}{\sqrt{1 - v^2/c^2}} \quad \therefore \quad L_0 = x_2 - x_1 = \frac{1}{\sqrt{1 - v^2/c^2}}\left[x_2' - x_1'\right]$$

$$\Rightarrow \quad L_0 = \frac{L}{\sqrt{1 - v^2/c^2}}$$

Therefore, in moving frame length is shorter than in rest frame.

## 9.7: Time dilation:

Time interval too is affected by relative motion. To an observer, a clock appears to run slower when in motion, than when at rest with respect to him.

Let we have two frame of reference, S and $S'$, the former at rest and latter moving with uniform velocity v, along +ve x-direction.

If two events occur at any given point $x'$ in frame $S'$, at time $t_1'$ and $t_2'$, as noted an the clock carried by it and at times $t_1$ and $t_2$, as noted on the clock carried by frame S, we clearly have time interval between two events, as noted in frame $S'$.

$$\Delta t' = (t_2' - t_1')$$

And, time interval between two events in frame S, $\Delta t = (t_2 - t_1)$

In accordance with inverse, Lorentz-transformation

$$t_1 = \frac{t_1' + v_x'/c^2}{\sqrt{1 - \frac{v^2}{c^2}}}; \quad t_2 = \frac{t_2' + v_x'/c^2}{\sqrt{1 - v^2/c^2}} \quad \therefore \quad \Delta t = (t_2 - t_1) = \frac{(t_2' - t_1')}{\sqrt{1 - v^2/c^2}}$$

$$\Rightarrow \quad \Delta t = \gamma \Delta t'$$

Since factor, $\gamma > 1$, we have $\Delta t > \Delta t'$

**Relativity of simultaneity :** The difference in time measurements in different inertial frames raises the question of re-examination of simultaneity of two different events in different frames.

Let two events occur at $x_1$ and $x_2$ in S-frame at times $t_1$ and $t_2$ and let them be simultaneous so that $t_1 = t_2$, but $x_1$ and $x_2$ should be different for otherwise the events would not appear as two different events.

If the $S'$ frame the events occur at times $t'_1$ and $t'_2$ respectively, then

$$t'_1 = \gamma\left(t_1 - vx_1/c^2\right); \; t'_2 = \gamma\left(t_2 - vx_2/c^2\right)$$

$\therefore \quad t'_1 \neq t'_2$ since $x_1 \neq x_2$

Thus, the events are not simultaneous in $S'$ frame. So, called simultaneity is also relative and the idea of a universal time is ruled out.

## 9.8: Velocity addition theorem:

Suppose a particles has a velocity 'u' in a frame of reference S at rest, such that its component along three coordinates are $u_x = \frac{dx}{dt}$, $u_y = \frac{dy}{dt}$, $u_z = \frac{dz}{dt}$, and its velocity in a frame of reference $S'$, moving with uniform velocity v relative to S along +ve direction of x-axis, is $u'$, with its component

$u'_x = \frac{dx'}{dt'}$, $u'_y = \frac{dy'}{dt'}$, $u'_z = \frac{dz'}{dt'}$ along three coordinates axes, as measured in $S'$.

Now, in accordance with Lorentz transformation:

$$x' = \gamma(x - vt);\; y' = y;\; z' = z;\; t' = \gamma\left(t - \frac{vx}{c^2}\right); \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$\Rightarrow \quad u'_x = \frac{dx'}{dt} = \frac{\gamma(dx - vdt)}{\gamma\left(dt - v\frac{dx}{c^2}\right)} = \frac{dx - vdt}{dt - v\frac{dx}{c^2}} \Rightarrow u'_x = \frac{dx/dt - v}{1 - \frac{vdx}{c^2dt}} = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

$$\Rightarrow \quad u'_y = \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - v\frac{dx}{c^2}\right)} = \frac{dy/dt}{\gamma\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)} \Rightarrow u'_y = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} \quad \Rightarrow u'_z = \frac{u_z}{\gamma\left(1 - \frac{vu_z}{c^2}\right)}$$

Therefore, the Lorentz velocity transformation is given by

$$\boxed{u'_x = \frac{u_x - v}{1 - u_x v/c^2};\; u'_y = \frac{u_y\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2};\; u'_z = \frac{u_z\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2}}$$

Conjugate equation

$$\boxed{u_x = \frac{u'_x - v}{1 + u'_x v/c^2};\; u_y = \frac{u'_y\sqrt{1 - v^2/c^2}}{1 + u'_x v/c^2};\; u_z = \frac{u'_z\sqrt{1 - v^2/c^2}}{1 + u'_x v/c^2}}$$

**Derivation of mass-variation relation:**

We shall now derive the mass-variation relation following the procedure of R.C. Tolman.

Consider two frames $S$ and $S'$ in uniform relative translation, parallel to $x$ (or $x'$)-axis, with velocity $v$. Let two balls of equal mass $m'$ in $S'$ frame collide elastically, moving along the $x$-axis with velocity $\mu'$ and $-\mu'$ in this frame and, on collision, coalesce together. At the moment of collision, their velocity is zero in $S'$ frame, but they still have a velocity $v$ in $S$ frame.

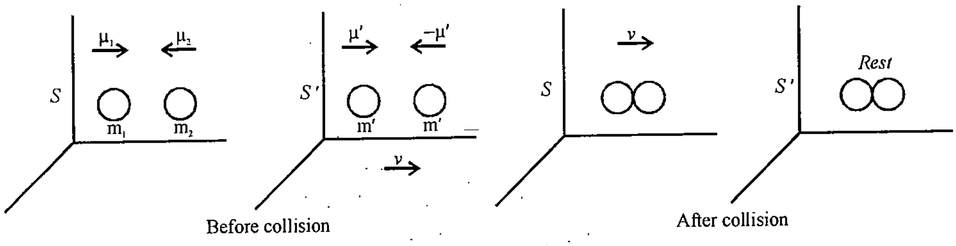


**Figure:** Collisions between two bodies as observed from two different frames

We shall describe the collision in $S$ and $S'$ frames. The conservation of momentum gives

$$m'\mu' - m'\mu' = 0 \qquad \text{in } S' \text{ frame} \qquad \ldots(1)$$

And $$m_1\mu_1 + m_2\mu_2 = (m_1 + m_2)v \qquad \text{in } S \text{ frame} \qquad \ldots(2)$$

By velocity addition theorem, $\mu_1$ and $\mu_2$, the velocities of masses in $S$ frame transform as

$$\mu_1 = \frac{\mu' + v}{1 + \mu' v/c^2}; \quad \mu_2 = \frac{-\mu' + v}{1 - \mu' v/c^2} \qquad \ldots(3)$$

Substituting these values of $\mu_1$ and $\mu_2$ in (2) and on simplification,

$$\frac{m_1}{m_2} = \frac{1 + \mu' v/c^2}{1 - \mu' v/c^2} = \frac{\sqrt{1 - \mu_2^2/c^2}}{1 - \mu_1^2/c^2} \qquad \ldots(4)$$

Simplification from equation (3), we may write

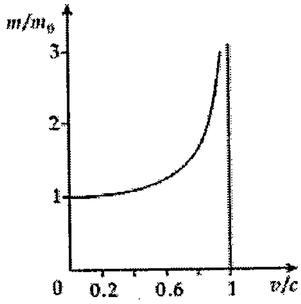
$$1 - \frac{\mu_1^2}{c^2} = 1 - \frac{(\mu'/c + v/c)^2}{(1 + \mu' v/c^2)^2} = \frac{(1 + \mu' v/c^2) - (\mu'/c + v/c)^2}{(1 + \mu' v/c^2)^2}$$

$$= \frac{(1 - v^2/c^2)(1 - \mu'^2/c^2)}{(1 + \mu' v/c^2)^2} = \frac{(1 - v^2/c^2)(1 - \mu'^2/c^2)}{(1 + \mu' v/c^2)^2}$$

Similarly, $$1 - \frac{\mu_2^2}{c^2} = \frac{(1 - v^2/c^2)(1 - \mu'^2/c^2)}{(1 - \mu' v/c^2)^2} \quad \therefore \frac{\sqrt{1 - \mu_2^2/c^2}}{\sqrt{1 - \mu_1^2/c^2}} = \frac{1 + \mu' v/c^2}{1 - \mu' v/c^2}$$

The two balls are assumed identical when at rest in the same frame. Let $m_0$ be the mass under this condition. By (4), the masses are inversely proportional to the quantity $\sqrt{1 - \mu^2/c^2}$, when they move with velocity $\mu$.

$$\therefore \quad m_1\sqrt{1 - \mu_1^2/c^2} = m_2\sqrt{1 - \mu_2^2/c^2} = m_0$$



where, $m_0$ is the rest mass of the body.
The mass in motion must, therefore, be

$$\boxed{m = \frac{m_0}{\sqrt{1 - v^2/c^2}}}$$

## Relativistic energy: Mass-energy equivalence:

We define the kinetic energy T of a particle in relativistic mechanics as the work done by an external force in increasing the velocity of the particle from zero to its present value $v$.

Considering, for simplicity, the one-dimensional motion of a particle in $x$-direction.

$$T = \int_0^x F\,dx$$

Where $F$ is the force-component along the displacement $dx$ and $x$ is the distance over which the force $F$ acts.

But $F = \frac{dp}{dt} = \frac{d}{dt}(mv)$

where, $m$ is the mass of the particle moving with velocity $v$.

$$\therefore \quad T = \int_0^x \frac{d}{dt}(mv)\,dx = \int_0^{mv} v\,d(mv) = \int_0^v v\,d\left(\frac{m_0 v}{\sqrt{1-v^2/c^2}}\right)$$

Integrating by parts, we get

$$T = \frac{m_0 v^2}{\sqrt{1-v^2/c^2}} - m_0 \int_0^v \frac{v\,dv}{\sqrt{1-v^2/c^2}}$$

$$= \frac{m_0 v^2}{\sqrt{1-v^2/c^2}} - m_0 \left| -c^2\sqrt{1-v^2/c^2} \right|_0^v$$

$$= \frac{m_0 c^2}{\sqrt{1-v^2/c^2}} - m_0 c^2 = mc^2 - m_0 c^2$$

$$\therefore \quad \boxed{T = mc^2 - m_0 c^2}$$

Total energy = kinetic energy + rest mass energy.

$$\therefore \quad E = T + m_0 c^2 = (m - m_0)c^2 + m_0 c^2 = mc^2$$

$$\boxed{E = mc^2}$$

**Total relativistic energy in terms of momentum:**

Sometimes, it is convenient and useful to express the total relativistic energy $E$ in terms of the momentum $p$. We have,

$$m^2c^4 - m_0^2c^4 = m_0^2c^4\left(\frac{1}{1-v^2/c^2} - 1\right)$$

$$= m_0^2c^4 \frac{v^2/c^2}{1-v^2/c^2} = c^2 \frac{m_0^2 v^2}{1-v^2/c^2} = c^2 p^2$$

$$\therefore \quad m^2c^4 = p^2c^2 + m_0^2c^4 \Rightarrow E^2 = p^2c^2 + m_0^2c^4$$

$$\therefore \quad \boxed{E = \sqrt{p^2c^2 + m_0^2c^4}}$$

**An important theorem:**

If $u$ and $u'$ be the velocities of a particle in $S$ and $S'$ frames respectively, then

$$\sqrt{1-\frac{u'^2}{c^2}}=\frac{\sqrt{\left(1+v^2/c^2\right)\left(1-u^2/c^2\right)}}{1-u_xv/c^2}$$

**Proof:** From the relativistic velocity addition theorem,

$$u'_x=\frac{v_x-v}{1-\frac{u_xv}{c^2}};\quad u'_y=\frac{u_y\sqrt{1-v^2/c^2}}{1-\frac{u_xv}{c^2}};\quad u'_z=\frac{u_z\sqrt{1-v^2/c^2}}{1-\frac{u_xv}{c^2}} \qquad \ldots(1)$$

where, $u_x$'s and $u'_x$'s are the component velocities.

$$\therefore \quad u'^2=u'^2_x+u'^2_y+u'^2_z=\frac{(u_x-v)^2+\left(u_y^2+u_z^2\right)\left(1-v^2/c^2\right)}{\left(1-u_xv/c^2\right)^2}\text{, using (1)}$$

$$\therefore \quad \frac{u'^2}{c^2}=\frac{\left(\frac{u_x}{c}-\frac{v}{c}\right)^2+\left(\frac{u_x}{c}-\frac{u_x^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1-u_xv/c^2\right)^2} \qquad \left[\because\ u_y^2+u_z^2=u^2-u_x^2\right]$$

$$\Rightarrow \quad 1-\frac{u'^2}{c^2}=\frac{\left(1-\frac{u_xv}{c^2}\right)-\left(\frac{u_x}{c}-\frac{v}{c}\right)^2-\left(\frac{u^2}{c^2}-\frac{u_x^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1-u_xv/c^2\right)^2}$$

$$=\frac{\left(1-\frac{v^2}{c^2}\right)-\frac{u^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}{\left(1-u_xv/c^2\right)^2}=\frac{\left(1-v^2/c^2\right)\left(1-u^2/c^2\right)}{\left(1-u_xv/c^2\right)^2} \qquad \ldots(2)$$

$$\boxed{1-\frac{u'^2}{c^2}=\frac{\left(1-v^2/c^2\right)\left(1-u^2/c^2\right)}{\left(1-u_xv/c^2\right)^2}}$$

**Transformation formulae for momentum, energy and force:**

Momentum in $S$ frame, the components of momentum are

$$p_x=mu_x=\frac{m_0u_x}{\sqrt{1-u^2/c^2}};\quad p_y=\frac{m_0v_y}{\sqrt{1-u^2/c^2}};\quad p_z=\frac{m_0u_z}{\sqrt{1-u^2/c^2}}$$

where, $u_x$'s are the component velocities.

If $p'_x$, $p'_y$ and $p'_z$ be the momentum components in $S'$ frame

$$p'_x = \frac{m_0 u'_x}{\sqrt{1-u'^2/c^2}} = \frac{m_0\left(1-u_x v/c^2\right)}{\sqrt{\left(1-v^2/c^2\right)\left(1-u^2/c^2\right)}} \times \frac{u_x - v}{1-u_x v/c^2}$$

$$= \frac{m_0 u_x - m_0 v}{\sqrt{1-v^2/c^2}\sqrt{1-u^2/c^2}} - \frac{m_0 c^2\left(v/c^2\right)}{\sqrt{\left(1-v^2/c^2\right)\left(1-u^2/c^2\right)}}$$

$$= \frac{p_x - \left(v/c^2\right)E}{\sqrt{1-v^2/c^2}} \quad \left(\because\ E = mc^2 = m_0 c^2/\sqrt{1-u^2/c^2}\right)$$

Also, $$p'_y = \frac{m_0 u'_y}{\sqrt{1-u'^2/c^2}} = \frac{m_0}{\sqrt{1-u'^2/c^2}} \times \frac{u_y\sqrt{1-v^2/c^2}}{\left(1-u_x v/c^2\right)}$$

$$= \frac{m_0\left(1-v_x v/c^2\right)}{\sqrt{\left(1-u^2/c^2\right)\left(1-v^2/c^2\right)}} \times \frac{u_y\sqrt{1-v^2/c^2}}{1-u_x v/c^2}$$

$$= \frac{m_0 u_y}{\sqrt{1-u^2/c^2}} = p_y$$, similarly, we shall have $p'_z = p_z$

So, the transformation formulae for the momentum are

$$\boxed{p'_x = \frac{p_x - vE/c^2}{\sqrt{1-v^2/c^2}};\ p'_y = p_y;\ p'_z = p_z}$$

**Energy :** The energy $E'$ of a particle of mass $m'$ is $S'$-frame is

$$E' = m'c^2 = \frac{m_0 c^2}{\sqrt{1-u'^2/c^2}} = \frac{m_0 c^2\left(1-u_x v/c^2\right)}{\sqrt{\left(1-v^2/c^2\right)\left(1-u^2/c^2\right)}}$$

$$= \frac{mc^2}{\sqrt{1-v^2/c^2}} - \frac{m u_x}{\sqrt{1-v^2/c^2}} v = \frac{E - v p_x}{\sqrt{1-v^2/c^2}}$$

$$\therefore \quad \boxed{E' = \frac{E - v p_x}{\sqrt{1-v^2/c^2}}}$$

The conjugate transformation relation is

$$\boxed{E = \frac{E' + vp'_x}{\sqrt{1 - v^2/c^2}}}$$

**Force:** Let a particle of mass $m$, be under the action of force $\vec{F}$ and have a momentum $\vec{p}$ in an interial frame $S$. Then, $\vec{F} = \frac{d\vec{p}}{dt}$

$$\therefore \qquad F_x = \frac{dp_x}{dt};\; F_y = \frac{dp_y}{dt};\; F_z = \frac{dp_z}{dt}$$

where $F_x$'s and $p_x$'s are the components of force and momentum respectively.

The corresponding values in $S'$ frame moving relative to $S$ frame with a uniform velocity $v$ along $x$ (or $x'$)-axis are

$$F'_x = \frac{dp'_x}{dt'};\; F'_y = \frac{dp'_y}{dt'};\; F'_z = \frac{dp'_z}{dt'}$$

But the transformation equations for momentum give

$$p'_x = \gamma\left(p_x - vE/c^2\right);\; p'_y = p_y;\; p'_z = p_z;\ \text{also, we have } t' = \gamma\left(t - vx/c^2\right)$$

where, $\gamma = 1/\sqrt{1 - v^2/c^2}$

$$\therefore \qquad F'_x = \frac{dp'_x}{dt'} = \frac{dp'_x}{dt}\cdot\frac{dt}{dt'} = \frac{\frac{dp_x}{dt} - \frac{v}{c^2}\frac{dE}{dt}}{\sqrt{1 - v^2/c^2}}\cdot\frac{\sqrt{1 - v^2/c^2}}{1 - \frac{v}{c^2}\frac{dx}{dt}}$$

$$= \frac{F_x - \frac{v}{c^2}\frac{dE}{dt}}{1 - \frac{v}{c^2}u_x} \cdot \qquad \left(\because\ dx/dt = u_x\right)$$

But, from Dirac equation,

$$E^2 = p^2c^2 + m_0^2c^4 = \left(\vec{p}\cdot\vec{p}\right)c^2 + m_0^2c^4$$

$$\Rightarrow \qquad 2E\left(dE/dt\right) = 2\vec{p}\cdot\left(d\vec{p}/dt\right)c^2 = 2mc^2\left(\vec{u}\cdot\vec{F}\right) = 2E\left(\vec{u}\cdot\vec{F}\right)$$

$$\left(\because\ \vec{p} = m\vec{u};\ \vec{F} = d\vec{p}/dt\right)$$

$$\therefore \qquad \frac{dE}{dt} = \vec{u}\cdot\vec{F} = u_xF_x + u_yF_y + u_zF_z$$

$$\therefore \quad F'_x = \frac{F_x - (u_xF_x + u_yF_y + u_zF_z)v/c^2}{1 - u_xv/c^2}$$

$$= \frac{F_x(1 - u_xv/c^2) - v/c^2(v_yF_y + u_zF_z)}{1 - u_xv/c^2}$$

$$\boxed{F'_x = F_x - \frac{v/c^2}{1 - u_xv/c^2}(u_yF_y + u_zF_z)}$$

Again, $\boxed{F'_y = \frac{dp'_y}{dt'} = \frac{dp'_y}{dt}\frac{dt}{dt'} = \frac{dp_y}{dt}\frac{\sqrt{1 - v^2/c^2}}{1 - u_xv/c^2} = \frac{F_y\sqrt{1 - v^2/c^2}}{1 - u_xv/c^2}}$

Similarly, $\boxed{F'_z = \frac{F_z\sqrt{1 - v^2/c^2}}{1 - u_xv/c^2}}$

### Time-like and space-like intervals:

Let two events occur in S-frame at $(x, y, z, t)$ and at $(x + \Delta x, y, , z, t + \Delta t)$ respectively. If the space-separation $\Delta x$ between the two events is less than the distance that light travels in vacuum during the time-separation $\Delta t$, i.e., if $\Delta x - c\Delta t < 0$, it is called time-like interval, and if $\Delta x - c\Delta t > 0$, it is called space-like interval.

## SOLVED PROBLEMS

1. The period of a pendulum is measured to be 3.0 s in the reference frame of the pendulum. What is the period when measured by an observer moving at a speed of 0.95c relative to the pendulum?

**Soln:** Instead of the observer moving at 0.95c, we can take the equivalent point of view that the observer is at rest and the pendulum is moving at 0.95c past the stationary observer. Hence, the pendulum is an example of a moving clock.

The proper time is $\Delta t_p = 3.0\ s$. Because a moving clock runs more slowly than a stationary clock by a factor of

$$\Delta t = \gamma \Delta t_p = \frac{1}{\sqrt{1-\frac{(0.95\,c)^2}{c^2}}}\Delta t_p = \frac{1}{\sqrt{1-0.902}}\Delta t_p = (3.2)(3.0\ s) = 9.6\ s$$

That is, a moving pendulum takes longer to complete a period than a pendulum at rest does.

2. Suppose you are driving your car on a business trip and are traveling at 30 m/s. Your boss, who is waiting at your destination, expects the trip to take 5.0 h. When you arrive late, your excuse is that your car clock registered the passage of 5.0 h but that you were driving fast and so your clock ran more slowly than your boss's clock. If your car clock actually did indicate a 5.0-h trip, how much time passed on your boss's clock, which was at rest on the Earth?:

**Soln:** Here, $\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-\frac{(3\times10^1\ m/s)^2}{(3\times10^8\ m/s)^2}}} = \frac{1}{\sqrt{1-10^{-14}}}$

Perform binomial expansion, we get

$$\gamma = \left(1-10^{-14}\right)^{-1/2} \approx 1+\frac{1}{9}\left(10^{-14}\right) = 1+5.0\times10^{-15}$$

In daily life, $\gamma$ is not much different from 1

$$\Delta t = \gamma\,\Delta t_p = \left(1+5.0\times10^{-15}\right)(5.0\,h) = 5.0\,h + 2.5\times10^{-14}\,h = \boxed{5.0h + 0.09\ ns}$$

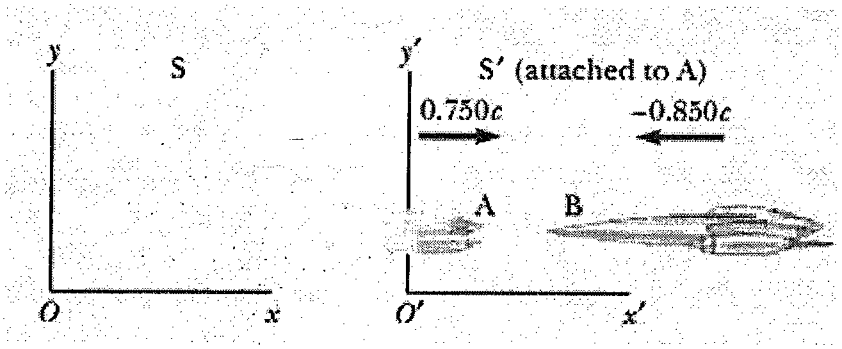
Your boss's clock would be only 0.09 ns ahead of your car clock. You might want to try another excuse!

3. A spaceship is measured to be 120.0 m long and 20.0 m in diameter while at rest relative to an observer. If this spaceship now flies by the observer with a speed of 0.99c, what length and diameter does the observer measure?

**Soln:** $L = L_p\sqrt{1-v^2/c^2} = (120.0\ m)\sqrt{1-\frac{(0.99\,c^2)}{c^2}} = \boxed{17\ m}$

The diameter measured by the observer is still 20.0 m because the diameter is a dimension perpendicular to the motion and length contraction occurs only along the direction of motion.

4. Two spaceships A and B are moving in opposite directions. An observer on the Earth measures the speed of ship A to be 0.750c and the speed of ship B to be 0.850c. Find the velocity of ship B as observed by the crew on ship A?



**Soln:** We can solve this problem by taking the S' frame as being attached to ship A, so that relative to the Earth (the S frame). We can consider ship B as moving with a velocity relative to the Earth. Hence, we can obtain the velocity of ship B relative to ship A by using velocity addition formula

$$u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{-0.850\,c - 0.75\,c}{1 - \frac{(-0.850\,c)(0.750\,c)}{c^2}} = \boxed{-0.977\,c}$$

The negative sign indicates that ship B is moving in the negative x direction as observed by the crew on ship A.

5. Imagine a motorcycle moving with a speed 0.80c past a stationary observer. If the rider tosses a ball in the forward direction with a speed of 0.70c relative to himself, what is the speed of the ball relative to the stationary observer?

**Soln:** The speed of the motorcycle relative to the stationary observer is $v = 0.8c$. The speed of the ball in the frame of reference of the motorcyclist is $u_x' = 0.7c$. Therefore, the speed $u_x$ of the ball relative to the stationary observer is

$$u_x = \frac{u_x' + v}{1 + \frac{u_x' v}{c^2}} = \frac{0.70\,c + 0.80\,c}{1 + \frac{(0.70\,c)(0.80\,c)}{c^2}} = \boxed{0.96\,c}$$

6. An electron, which has a mass of $9.11 \times 10^{31}$ kg, moves with a speed of 0.750c. Find its relativistic momentum.

**Soln:** $$p = \frac{m_e u}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{(9.11 \times 10^{-31}\,kg)(0.750 \times 3.00 \times 10^8\,m/s)}{\sqrt{1 - \frac{(0.750\,c)^2}{c^2}}} = \boxed{3.10 \times 10^{-22}\,kg\,m/s}$$

7. An electron in a television picture tube typically moves with a speed Find its total energy and kinetic energy in electron volts.

**Soln:** Rest energy of electron is 0.511 MeV

$$E = \frac{m_e c^2}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{0.511\,MeV}{\sqrt{1 - \frac{(0.250\,c)^2}{c^2}}} = 1.03(0.511\,MeV) = 0.528\,MeV$$

This is 3% greater than the rest energy.

We obtain the kinetic energy by subtracting the rest energy from the total energy:

$$K = E - m_e c^2 = 0.528\,MeV - 0.511\,MeV = \boxed{0.017\,MeV}$$

3. (a) Find the rest energy of a proton in electron volts.
(b) If the total energy of a proton is three times its rest energy, with what speed is the proton moving?
(c) Determine the kinetic energy of the proton in electron volts.
(d) What is the proton's momentum?

**Soln:** (a) $E_R = m_p c^2 = \left(1.67\times10^{-27}\ kg\right)\left(3.00\times10^{8}\ m/s\right)^2$

$$= \left(1.50\times10^{-10}\ J\right)\left(1.00\,eV/1.60\times10^{-19}\ J\right) = \boxed{938\ MeV}$$

(b) $E = 3m_p c^2 = \dfrac{m_p c^2}{\sqrt{1-\dfrac{u^2}{c^2}}} \qquad \Rightarrow 3 = \dfrac{1}{\sqrt{1-\dfrac{u^2}{c^2}}}$

Solving for 'u' gives

$$1-\frac{u^2}{c^2}=\frac{1}{9} \quad \Rightarrow \quad \frac{u^2}{c^2}=\frac{8}{9} \quad \Rightarrow u=\frac{\sqrt{3}}{8}c=\boxed{2.83\times10^{8}\ m/s}$$

(c) $K = E - m_p c^2 = 3m_p c^2 - m_p c^2 = 2m_p c^2$

Because, $m_p c^2 = 938\,MeV$, $K = \boxed{1.880\ MeV}$

(d) $E^2 = p^2c^2 + \left(m_p c^2\right)^2 = \left(3\,m_p c^2\right)^2 \Rightarrow p^2c^2 = 9\left(m_p c^2\right)^2 - \left(m_p c^2\right)^2 = 8\left(m_p c^2\right)^2$

$$\Rightarrow p = \sqrt{8}\,\frac{m_p c^2}{c} = \sqrt{8}\,\frac{(938\ MeV)}{c} = \boxed{2\,650\ MeV/c}$$

9. A deuteron, which is the nucleus of a deuterium atom, contains one proton and one neutron and has a mass of 2.013 553 u. This total deuteron mass is not equal to the sum of the masses of the proton and neutron. Calculate the mass difference and determine its energy equivalence, which is called the binding energy of the nucleus.

**Soln:** Using atomic mass units (u), we have

$m_p$ = mass of proton = 1.007 276 u

$m_n$ = mass of neutron = 1.008 665 u

$$m_p + m_n = 2.015\,941\,u$$

The mass difference $\Delta m$ is therefore 0.002 388 u

By definition; 1 u = $1.66\times10^{-27}$ kg

$$\Delta m = 0.002\,388\,u = \boxed{3.96\times10^{-30}\ kg}$$

Using $E = \Delta mc^2$, we find that the binding energy is:

$$E = \Delta mc^2 = \left(3.96\times10^{-30}\ kg\right)\left(3.00\times10^{8}\ m/s\right)^2 = 3.56\times10^{-13}\ J = \boxed{2.23\ MeV}$$

Therefore, the minimum energy required to separate the proton from the neutron of the deuterium nucleus (the binding energy) is 2.23 MeV.

10. If $(x, y, z, t)$ be an event in *S*-system and $(x', y', z', t')$ be the same event in $S'$-system that moves relative to *S*-system with a uniform velocity *v* along *x*-direction, show that $ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$ is invariant under Lorentz transformation.

**Soln.** Using Lorentz transformation in differential form, $dx = \gamma\left(dx' + vdt'\right)$; $dy = dy'$; $dz = dz'$; $dt = \gamma\left[dt' + \left(v/c^2\right)dx'\right]$ where *v* is constant and $\gamma$ =Lorentz factor.

$$\therefore \quad ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$$

$$=\gamma^2\left(dx'+vdt'\right)^2+d'y^2+dz'^2-c^2\gamma^2\left(dt'+\left(v/c^2\right)dx'\right)^2$$

$$=\gamma^2\left(dx'^2+v^2dt'^2+2vdx'dt'\right)-\gamma^2c^2\left(dt'^2+2\frac{v}{c^2}dx'dt'\right)+dy'^2+dx'^2$$

$$=\gamma^2dx'^2\left(1-\frac{v^2}{c^2}\right)+dy'^2+dz'^2-\gamma^2c^2dt'^2\left(1-\frac{v^2}{c^2}\right)$$

$$=dx'^2+dy'^2+dz'^2-c^2dt'^2 \qquad \left(\because\ \gamma=1/\sqrt{1-v^2/c^2}\right)$$

$$=ds'^2$$

Therefore, $ds^2$ is invariant under Lorentz transformation.

11. A rigid bar of length 1.5m is at rest relative to frame $S'$. If it makes an angle $\alpha'=45^\circ$ with $x'-axis$, find the length of the bar and its orientation relative to frame $S$, when $v=0.95c$?

**Soln.** Resolving the length of the bar into components parallel to $x'$ and $y'$ axes, the corresponding length in $S'$ are:

$$L'_x=L'\cos\alpha'=1.5\times1/\sqrt{2}=1.06m;\ L'_y=L'\sin\alpha'=1.5\times1/\sqrt{2}=1.06\,m$$

The vertical components would show no contraction, when viewed from $S$. Horizontal component, being parallel to $v$, would

$$\therefore \qquad L_y=L'_y=1.06\,m$$

$$L_x=L'_x\sqrt{1-v^2/c^2}=1.06\sqrt{1-0.95^2}=0.331\,m$$

Therefore, length of the relative $S$ is:

$$L_1=\sqrt{L_x^2+L_y^2}=\sqrt{(1.06)^2+(0.331)^2}=1.11\,m$$

Orientation of the bar relative to $S$ is

$$\tan\alpha=\frac{L_y}{L_x}=\frac{1.06}{0.331}=3.20 \qquad \therefore\ \alpha=72^\circ 40'$$

12. Show that if $l_0^3$ is the rest volume of a cube, then $l_0^3\left(1-\beta^2\right)^{1/2}$ is the volume as viewed from a reference frame among moving with uniform velocity $v$ parallel to an edge of the cube, $\beta=v/c$

**Soln.** The rest length of the cube was $l_0$ at each side so that the rest volume was $l_0^3$. Now, along the direction of motion, the length $l_0$ contracts to $\ell_0\left(1-\beta^2\right)^{1/2}$. In the other two directions, there is no Lorentz contraction and the length remains $l_0$.

Therefore, moving volume of the cube $\ell_0\left(1-\beta^2\right)^{1/2}\times\ell_0\times\ell_0=\ell_0^3\left(1-\beta^2\right)^{1/2}$

3. An experimental intends to study a beam of $\pi$ -mesons of velocity $0.9c$. How far can he place his apparatus from the target where $\pi$ -measons are produced and still expect to get-sufficient number of $\pi$ -mesons.

**Soln.** With relativistic time dilation,

$$\Delta t = \Delta t' / \sqrt{1-0.9^2} = 2.3\Delta t'$$

Since $\Delta t'$ is the time-interval measured by an observer moving with $\pi$ -mesons, it is the real life-time, so that $\Delta t' = 2\times10^{-8}s$. So, in the lab-frame, the mean life is

$$\Delta t = 2.3\times\Delta t' = 2.3\times2\times10^{-8}s = 4.6\times10^{-8}s$$

Therefore, average distance travelled by $\pi$ -mesons $= v.\Delta t = 0.9\times3\times10^{8}\times4.6\times10^{-8}m = 12.42\ m$

14. The half-life of a particular particle as measured in the lab is $4.0\times10^{-8}s$ when its speed is 0.80c and $3\times10^{-8}s$ when its speed is 0.60c. Find its actual life time.

**Soln.** Due to time-dilation, the half-life of a moving particle appears lengthened. Let $\Delta t_0$ be the actual half-life, i.e., in the frame of reference of the particle, and $\Delta t$ that measured by a stationary observer.

$$\therefore \quad \Delta t = \Delta t_0 / \sqrt{1-v^2/c^2}$$

Therefore, by the problem:

$$4.0\times10^{-8} = \frac{\Delta t}{\sqrt{1-0.8^2}};\ 3.0\times10^{-8} = \frac{\Delta t_0}{\sqrt{1-0.6^2}}$$

Either relation gives, $\Delta t_0 = 2.4\times10^{-8}s$, the actual life-time.

15. What will be the period of the 'seconds' pendulum measured by an observer moving with a speed of 0.8c?

**Soln.** If $T_0\ (=2s)$ be the period of a seconds pendulum as measured by an observer at rest, the same would be, say T to the moving observer.

Therefore, from the time-dilation relation,

$$T = \frac{T_0}{\sqrt{1-v^2/c^2}} = \frac{2}{\sqrt{1-0.8^2}} = 3.33s$$

16. The average life-time of neutron is 15 min. It disintegrates spontaneously into an electron, a proton and a neutrino. If the distance of the sun from the earth is $11\times10^{10}m$, find the average minimum velocity with which a neutron must leave the sun so as to reach the earth before breaking up.

**Soln.** Let $v$ be the required velocity of the neutron. The time $t_1$ for neutron to reach the earth from the sun before decay is

$$t_1 = \frac{11\times10^{10}m}{v}$$

If $t_0$ be the life-time of neutron at rest, then the average life-time $t$ of the 'moving' neutro, as measured by an observer on earth, is

$$t = \frac{t_0}{\sqrt{1-v^2/c^2}} = \frac{15\times60s}{\sqrt{1-v^2/c^2}}$$

Plainly, this $t$ and previous $t_1$ must be equal,

$$\therefore \quad \frac{11\times10^{10}}{v} = \frac{15\times60}{\sqrt{1-v^2/\left(3\times10^8\right)^2}}$$

Simplifying, we obtain,

$$v = 1.132\times10^{8}\ ms^{-1}$$

17. A rocket is travelling towards the moon with a velocity 0.6c. When half-way to the moon, it fires a message rocket back towards the earth with a velocity 0.8c relative to the primary rocket. What a velocity of the message rocket as seen by a terrestrial observer? Neglect the recoil.

**Soln.** The velocity of the message rocket relative to a terrestrial observer is, by velocity addition theorem,

$$v=\frac{(0.6-0.8)c}{1-0.8\times0.6}=\frac{-0.2c}{1-0.48}=-\frac{0.2c}{0.52}=-0.385c$$

The negative sign indicates that the rocket is moving towards the earth.

18. The speed of an electron in an uniform electric field changes from 0.95c to 0.98c. Find the change in mass and the workd one on the electron to change the velocity.

**Soln.** Let $m_1$, $m_2$ be the masses of the electron corresponding to velocities 0.95c and 0.98c.

$$\therefore \quad m_1=\frac{m_0}{\sqrt{1-0.95^2}}=3.2\,m_0;\ m_2=\frac{m_0}{\sqrt{1-0.98^2}}=5.0\,m_0$$

where $m_0$ = rest mass of electron $=9.11\times10^{-31}kg$

Therefore, change in mass,

$$\Delta m=m_2-m_1=(5.0-3.2)m_0$$

$$=1.8\times9.11\times10^{-31}kg=16.4\times10^{-31}kg$$

Work done on electron = change in its kinetic energy.

$$\Delta T=T_2-T_1=(\Delta m)c^2=1.8m_0c^2=1.8\times0.51\,MeV=0.92\,MeV$$

19. An electron of rest mass $9.1\times10^{-31}kg$ is moving with a speed 0.99c. What is its total energy? Find theratio of Newtonian kinetic energy to the relativistic kinetic energy.

**Soln.** The mass of electron corresponding to velocity v = 099c is

$$m=\frac{m_0}{\sqrt{1-0.99^2}}=\frac{9.1\times10^{-31}}{0.141}=64.54\times10^{-31}kg$$

Therefore, total energy of electron,

$$E=mc^2=(64.54\times10^{-31})\times(3\times10^8)^2=5.81\times10^{-13}J$$

$$\frac{(K.E.)_{Newton}}{(K.E.)_{relativistic}}=\frac{\frac{1}{2}m_0v^2}{mc^2-m_0c^2}=\frac{1}{2}\frac{m_0}{(m-m_0)}\left(\frac{v}{c}\right)^2=\frac{9.1}{2(64.54-9.10)}(0.99)^2=0.08$$

0. The rest mass of electron is $9.11\times10^{-31}kg$. What is its momentum (relativistic) when it is moving down the axis of a linear accelerator tube of a speed of 0.4c relative to the accelertor tube?

**oln.** The required relativistc momentum of the electron is

$$p=\frac{m_0v}{\sqrt{1-v^2/c^2}}=\frac{9.11\times10^{-31}\times0.4c}{\sqrt{1-0.4^2}}=1.19\times10^{-22}kg.m/s$$

1. A certain quantity of ice at 0°C melts into water at 0°C and thereby gains 1 kg mass. Find its initial mass.

**oln.** $1kg$ mass $=1\times(3\times10^8)^2\,J$ energy. Let $m$ be the initial mass of ice in kg.

Gain in energy per kg. of ice at 0°C melting into water at 0°C is :

$$80\times1000\,calorie=80\times1000\times4.181J$$

Therefore, total energy gain $=80\times1000\times4.18\times m\ joule$

Therefore, by the problem, $80\times 1000\times 4.18\times m=\left(3\times 10^{8}\right)^{2}$

Simplifying, $m=2.69\times 10^{11}\ kg$

22. A star of radius $r$ contracts to a radius $r'$ while its temperature remains unchanged. Find an expression for the energy radiated by the star, assuming uniform density in each case.

**Soln.** The density of the star of mass $m$ is $\rho=m/\frac{4}{3}\pi r^{3}$. The new mass $m'$, when the star contacts to a radius $r'$, is

$$m'=\frac{4}{3}\pi r'^{3}\rho=\frac{4}{3}\pi r'^{3}\cdot\frac{m}{\frac{4}{3}\pi r^{3}}=m\left(r'/r\right)^{3}$$

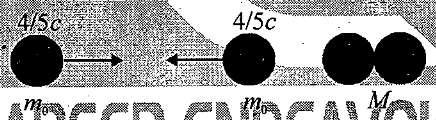
Therefore, mass loss due to contraction, $\Delta m=m-m'=m\left[1-\left(r'/r\right)^{3}\right]$

Therefore, energy radiated by the star $=\Delta m\cdot c^{2}=mc^{2}\left[1-\left(r'/r\right)^{3}\right]$

23. Two lumps of clay, each of first mass $m_0$, collide head on at $\frac{4}{5}c$ and stick together. Calculate the mass of the composite lump. Explain the result obtained.

**Soln.** Let M be the mass of composite lump. The conservation of momentum is already satisfied. : zero before collision, zero after collision.

The energy of each lump before collision is $m_0c^2/\sqrt{1-(4/5)^2}=\frac{5}{3}m_0c^2$. The energy of composite lump after collision is $Mc^2$. So, from conservation of energy principle,



$$Mc^{2}=2\times\frac{5}{3}m_{0}c^{2}\ \Rightarrow M=\frac{10}{3}m_{0}$$

The result shows that the mass of the composite lump is more than the sum of initial masses. Thus, in the collision mass is not conserved; the mass-increase is due to the kinetic energy being converted into rest energy.

24. A particle leaving a cyclotron has relativistic energy 10 GeV and a relativistic momentum 8 GeV/C. What is the rest mass of particle? **[H.C.U.-2014]**

(a) 0.25 GeV/C$^2$ (b) 1.20 GeV/C$^2$ (c) 2.00 GeV/C$^2$ (d) 6.00 GeV/C$^2$

**Soln.** $E=10\ \text{GeV}$; $p=8\ \text{GeV/c}$

Relationship between relativistic energy, momentum and mass is given by,

$$E=\sqrt{p^{2}c^{2}+m_{0}^{2}c^{4}}$$

$$\Rightarrow\ 10=\sqrt{\frac{8^{2}}{c^{2}}\cdot c^{2}+m_{0}^{2}c^{4}}\ \Rightarrow\ 10^{2}=8^{2}+m_{0}^{2}c^{4}$$

$$\Rightarrow\ m_{0}^{2}c^{4}=10^{2}-8^{2}=36\ \Rightarrow\ m_{0}c^{2}=6\ \text{GeV}$$

$$\Rightarrow\ m_{0}=6\ \text{GeV}/c^{2}$$

**Correct option is (d)**

25. If the total energy of a particle is thrice its rest mass energy, then the velocity of particle in terms of velocity of light $c$ is

(a) $\frac{c}{3}$ (b) $\frac{2c}{3}$ (c) $\frac{2\sqrt{2}}{3}c$ (d) $\frac{\sqrt{2}}{3}c$ **[BHU 2010]**

**Soln.** The total energy $E$ is $E = mc^2$

$$\Rightarrow 3m_0c^2 = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} \Rightarrow 1-\frac{v^2}{c^2} = \frac{1}{9} \Rightarrow v = \frac{2\sqrt{2}}{3}c.$$

**Correct option is (c)**

26. The kinetic energy of an electron moving with speed of 0.6 $c$ and rest mass energy 0.511 MeV, is **[BHU 2010]**

(a) 0.224 MeV (b) 0.128 MeV (c) 0.64 MeV (d) 0.540 MeV

**Soln.** The kinetic energy is

$$\text{K.E.} = (m-m_0)c^2 = \left(\frac{m_0}{\sqrt{1-v^2/c^2}} - m_0\right)c^2$$

$$= m_0c^2\left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right)$$

$$= 0.511\left(\frac{1}{\sqrt{1-0.36}} - 1\right)$$

$$= \frac{0.511}{4} \simeq 0.128 \text{ MeV}.$$

**Correct option is (b)**

27. Two successive Lorentz transformation with velocities $v_1$ and $v_2$ in same direction are equivalent to a single Lorentz transformation with a velocity **[BHU 2010]**

(a) $v_1 + v_2$ (b) $\frac{v_1+v_2}{1+\frac{v_1v_2}{c^2}}$ (c) $\frac{v_1+v_2}{1-\frac{v_1v_2}{c^2}}$ (d) $\frac{v_1-v_2}{1-\frac{v_1v_2}{c^2}}$

**Soln.** The single Lorentz transformation velocity or the relative velocity is

$$v = \frac{v_1 - v_2}{1 - \frac{v_1v_2}{c^2}}$$

**Correct option is (d)**

28. For which value of $\frac{v}{c}$ for a particle will the relativistic mass of the particle exceed its rest mass by a given fraction $f$? **[BHU 2011]**

(a) $\frac{\sqrt{f(1+f)}}{(2+f)}$ (b) $\frac{\sqrt{f(2+f)}}{(1+f)}$ (c) $\frac{\sqrt{2f(1+f)}}{(2+f)}$ (d) $\frac{\sqrt{f(2+f)}}{2(1+f)}$

**Soln.** GIven: $\frac{m-m_0}{m_0} = f \Rightarrow \sqrt{\frac{1}{1-\frac{v^2}{c^2}}} - 1 = f \Rightarrow 1-\frac{v^2}{c^2} = \frac{1}{(1+f)^2}$

$$\Rightarrow \frac{v^2}{c^2} = 1 - \frac{1}{(1+f)^2} = \frac{f^2+2f}{\left(1+f^2\right)} \quad \Rightarrow \frac{v}{c} = \frac{\sqrt{f(f+2)}}{1+f}$$

**Correct option is (b)**

29. The velocity of an object whose length appears to be contracted to half of its proper length is

**[BHU 2011]**

(a) $\frac{1}{\sqrt{2}}c$ (b) $\frac{2}{\sqrt{3}}c$ (c) $\frac{1}{2}c$ (d) $\frac{\sqrt{3}}{2}c$

**Soln.** Given: $l = l_0\sqrt{1-\frac{v^2}{c^2}} = \frac{l_0}{2} \Rightarrow \sqrt{1-\frac{v^2}{c^2}} = \frac{1}{2} \Rightarrow v = \frac{\sqrt{3}}{2}c$

**Correct option is (d)**

30. If the total energy of the particle is thrice its rest mass energy, then the velocity of the particle is

**[BHU 2011]**

(a) $\frac{c}{3}$ (b) $\frac{2c}{3}$ (c) $\frac{\sqrt{2}}{3}c$ (d) $\frac{2\sqrt{2}}{3}c$

**Soln.** The total energy $E$ is $E = mc^2$

$$\Rightarrow 3m_0c^2 = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} \Rightarrow 1-\frac{v^2}{c^2} = \frac{1}{9} \Rightarrow v = \frac{2\sqrt{2}}{3}c.$$

**Correct option is (d)**

31. If the proper mean lifetime $\tau$ of $\pi^+$ meson is $2.5\times10^{-8}$ sec, then the distance travelled by a burst of $\pi^+$ mesons travelling with v = 0.73c is

**[BHU 2011]**

(a) 800 cm (b) 600 cm (c) 500 cm (d) 300 cm

**Soln.** Lifetime of meson measured in Lab frame

$$t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2.5\times10^{-8}}{\sqrt{1-(0.73)^2}}s = 3.68\times10^{-8}\ s$$

Distance travelled by the mesons will be

$$d = \left(3.68\times10^{-8}\right)\times\left(0.73\times3\times10^{8}\right)m = 8.05\ m$$

**Correct option is (a)**

32. If the total energy of a particle of mass $m$ is equal to twice its rest energy, then the magnitude of the particle relativistic mechanism is **[BHU 2012]**

(a) $\frac{mc}{2}$ (b) $\frac{mc}{\sqrt{2}}$ (c) $mc$ (d) $\sqrt{3}\,mc$

**Soln.** The total energy is $E^2 = m^2c^4 + p^2c^2$

$\Rightarrow (2mc^2) = m^2c^4 + p^2c^2 \Rightarrow p^2c^2 = 3m^2c^4 \Rightarrow p = \sqrt{3}mc$

**Correct option is (d)**

33. If a charged pion that decays in $10^{-8}$ second in its own rest frame is to travel 30 meters in the laboratory before decaying, the pion's speed must be most nearly **[BHU 2012]**

(a) $0.43 \times 10^8$ m/s (b) $2.84 \times 10^8$ m/s (c) $2.90 \times 10^8$ m/s (d) $2.98 \times 10^8$ m/s

**Soln.** Given : $\tau_0 = 10^{-8}$ sec

So the time in the laboratory frame is

$$\tau = \frac{\tau_0}{\sqrt{1-v^2/c^2}} = \frac{10^{-8}}{\sqrt{1-v^2/c^2}}$$

$\therefore \quad L = \tau v$

$$\Rightarrow \quad 30 = \frac{10^{-8}}{\sqrt{1-v^2/c^2}}v$$

$$\Rightarrow \quad 900\left(1-\frac{v^2}{c^2}\right) = 10^{-16}v^2$$

$$\Rightarrow \quad 900 = \left(\frac{900}{c^2}+10^{-16}\right)v^2$$

$$\Rightarrow \quad 900 = \left(10^{-14}+10^{-16}\right)v^2$$

$$\Rightarrow \quad v^2 = \frac{900}{1.01\times10^{-14}}$$

$$\Rightarrow \quad v = 29.8\times10^7 \text{ m/s} = 2.98\times10^8 \text{ m/s.}$$

**Correct option is (d)**

34. A particle leaving a cyclotron has a total relativistic energy of 10 GeV and a relativistic momentum of 9 GeV/$c$. What is the rest mass of this particle ? **[BHU 2012]**

(a) 0.25 GeV/$c^2$ (b) 1.20 GeV/$c^2$ (c) 2.00 GeV/$c^2$ (d) 6.00 GeV/$c^2$

**Soln.** The energy momentum equation is $E^2 = p^2c^2 + m_0^2c^4$

$\Rightarrow (10)^2 = (9)^2 + m_0^2c^4$

$\Rightarrow m_0^2c^4 = 19$

$\Rightarrow m_0 = 4.35$ GeV/c$^2$

**Correct option is ( )**

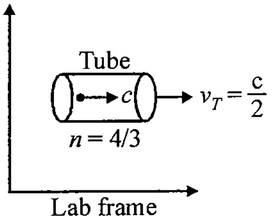
35. A tube of water is travelling at $\frac{1}{2}c$ relative to the lab frame when a beam of light travelling in the same direction as the tube enters it. What is the speed of light in the water relative to the lab frame ? (The index of refraction of water is 4/3). **[BHU 2012]**

(a) $\frac{1}{2c}$ (b) $\frac{2}{3c}$ (c) $\frac{5}{6c}$ (d) $\frac{10c}{11}$

**Soln.** The speed of light in water relative to the lab frame is

Given : $u_x = c$ ; $v_T = \frac{c}{2}$ ; $n = \frac{4}{3}$

The speed of light in the water relative to the lab frame is



$$u'_x = \frac{u_x/n + v_T}{1 + u_x v_T/nc^2}$$

$$\Rightarrow \quad u'_x = \frac{\frac{3c}{4} + \frac{c}{2}}{1 + \frac{3c^2}{24c^2}} = \frac{5c/4}{11/8} = \frac{10}{11}c.$$

**Correct option is (d)**

36. If the boost is along $x$-axis with a uniform velocity $\vec{v}$, the Lorentz invariant quantity is **[BHU 2013]**

(a) $y^2 - c^2t^2$ (b) $z^2 - c^2t^2$ (c) $y^2 + z^2 - c^2t^2$ (d) $x^2 - c^2t^2$

**Soln.** The Lorentz invariant quantity is $x^2 - c^2t^2$.

**Correct option is (d)**

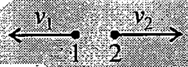
37. Two particles are travelling in the opposite directions with speed 0.9c relative to the laboratory frame. Their relative speed would be **[BHU 2013]**

(a) 0.995c (b) 1.8c (c) 0.895c (d) 0.905c

**Soln.** Given : $v_1 = v_2 = 0.9c$

The relative speed is

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} = \frac{1.8c}{1 + 0.81} = \frac{1.8}{1.81}c \simeq 0.99c.$$



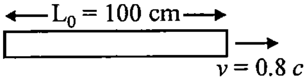
**Correct option is (a)**

38. The length of a rod moving with velocity equal to 0.8c, would be modified if the proper length is equal to 100 cm. The modified length of this moving rod would be equal to **[BHU 2013]**

(a) 50 cm (b) 60 cm (c) 70 cm (d) 65 cm

**Soln.** The modified length $L$ is

$$L = L_0\sqrt{1 - v^2/c^2}$$
$$= 100\sqrt{1 - (0.8)^2}$$
$$= 60 \text{ cm}$$



**Correct option is (b)**

39. If the boost is along $z$-axis with uniform velocity, the Lorentz invariant quantity would be **[BHU 2013]**

(a) $z^2 + c^2t^2$ (b) $z^2 + x^2$ (c) $z^2 + y^2$ (d) $x^2 + y^2$

**Soln.** The distance perpendicular to $z$-axis, i.e., $x^2 + y^2$ would be invariant.

**Correct option is (d)**

40. If $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\beta^2 = \frac{v^2}{c^2}$ it is clear that $\gamma^2 - \beta^2\gamma^2 = 1$. The choices of $\gamma$ and $\beta\gamma$ can be made in the language of trigonometrical functions as **[BHU 2013]**

(a) $\gamma = \operatorname{cosec}\theta,\ \beta\gamma = \cot v$ (b) $\gamma = \sec\theta,\ \beta\gamma = \tan\theta$

(c) $\gamma = \cosh\theta,\ \beta\gamma = \sinh\theta$ (d) None of the above

**Soln.** We choose $\frac{v}{c} = \tanh\theta$. It means, $\beta = \frac{v}{c} = \tanh\theta$.

It gives $\gamma^2(1-\beta^2) = 1$ as

$\Rightarrow\ \gamma = \cosh\theta$

$\therefore\ \beta\gamma = \sinh\theta$

**Correct option is (c)**

41. Which of the following remains invariant under Lorentz transformation ? **[BHU 2014]**

(a) charge density (b) current (c) charge (d) current density

**Soln.** The charge remains invariant under Lorentz transformation.

**Correct option is (c)**

42. The speed at which the kinetic energy of an electron is equal to twice its rest energy is : **[BHU 2014]**

(a) $\frac{2\sqrt{2}}{3}C$ (b) $\frac{\sqrt{2}}{3}C$ (c) $\frac{\sqrt{3}}{2}C$ (d) $\frac{C}{2}$

**Soln.** The kinetic energy is given by $\text{K.E.} = (m-m_0)c^2$, where $m$ and $m_0$ are the relativistic and the rest masses respectively.

As given, $\text{K.E.} = 2m_0c^2$

$\therefore\quad 2m_0c^2 = (m-m_0)c^2 \quad \Rightarrow\quad 3m_0c^2 = \frac{m_0}{\sqrt{1-v^2/c^2}}$

$\Rightarrow\quad 1-\frac{v^2}{c^2} = \frac{1}{9} \quad \Rightarrow\quad v = \frac{2\sqrt{2}}{3}c$

**Correct option is (a)**

43. A meter scale which is moving with a speed $v$ along its length appears to be centimeter scale to a stationary observer. Which of the following is correct ? **[BHU 2014]**

(a) $v^2 = 0.99\,c^2$ (b) $v^2 = 0.9999\,c^2$ (c) $v = 0.99\,c$ (d) $v^2 = 0.9\,c^2$

**Soln.** Given : $L_0 = 1\text{ m} = 100\text{ cm}$ and $L = 1\text{ cm}$

Using, $L = L_0\sqrt{1-v^2/c^2} \quad \Rightarrow\quad 1 = 100\sqrt{1-v^2/c^2}$

$\Rightarrow\quad \frac{1}{10000} = 1-\frac{v^2}{c^2} \quad \Rightarrow\quad \frac{v^2}{c^2} = \frac{9999}{10000} = 0.9999 \quad \Rightarrow\quad v^2 = 0.9999c^2$

**Correct option is (b)**

44. Kinetic energy of a free relativistic particle is given as ($m_0$ is rest mass, $p$ is the momentum of the particle):

(a) $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}m_0c^2$ (b) $pc$ (c) $\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}-1\right)m_0c^2$ (d) $mc^2$ **[BHU 2014]**

**Soln.** Kinetic energy is

$$\text{K.E.} = (m - m_0)c^2$$

$$= \left[\frac{m_0}{\left(1 - v^2/c^2\right)^{1/2}} - m_0\right]c^2 \qquad = \left[\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right]m_0 c^2$$

**Correct option is (c)**

45. Which of the following is incorrect ? **[BHU 2014]**
(a) Laws of mechanics are covariant under Galilean transformation
(b) Maxwell's equations are covariant under Galilean transformation
(c) Laws of mechanics are covariant under Lorentz transformation
(d) Maxwell's equations are covariant under Lorentz transformation

**Soln.** Maxwell's equations change their form under Lorentz transformation and hence are not covariant.

**Correct option is (d)**

46. Which of the following does not remain invariant in special theory of relativity ? **[BHU 2014]**

(a) $\frac{E^2}{c^2} - \vec{p}^2$ (b) $d^3\vec{p}$ (c) $\frac{d^3\vec{p}}{E}$ (d)

**Soln.** $d^3\vec{p} = dx\,dy\,dz$ is not invariant in special theory of relativity.

**Correct option is (b)**

47. If a clock moves with a very high velocity the time interval in that clock will appear to a stationary observer as:
(a) shorter (b) longer **[BHU 2014]**
(c) unchanged (d) shorter or longer on the direction of the velocity

**Soln.** The moving clock appears to be slower to a stationary observer.

**Correct option is (a)**

48. Which of the following statements is incorrect ? **[BHU 2014]**
(a) No signal can travel with velocity greater than C
(b) Simultanity is frame in dependent
(c) Proper time is same in all inertial frame
(d) Total energy of a particle does not depend on the choice of the inertial frame

**Soln.** Proper time depends on the velocity of the moving frame and hence depends on the inertial frame.

**Correct option is (c)**

49. Two photons A and B are moving in the opposite direction. The relative velocity of A w.r.t. B would be

(a) $2c$ (b) 0 (c) $c$ (d) $\frac{2}{3}c$ **[BHU 2014]**

**Soln.** Relative velocity of A w.r.t. B will be

$$v = \frac{v_1 - v_2}{1 - \frac{v_1 v_2}{c^2}} = \frac{c - (-c)}{1 - \frac{(c)(-c)}{c^2}} = c$$

**Correct option is (c)**

50. The inertial reference frame is used in the description of special theory of relativity. An example of an inertial reference frame is **[BHU 2015]**

(a) rotating frame with a uniform angular velocity w.r.t. a stationary frame
(b) two frames stationary with respect to each other
(c) A freely failing frame under uniform gravitation
(d) two frames having uniform acceleration with respect to each other

**Soln.** An inertial reference frame is either at rest or moving with constant velocity with respect to another frame.

**Correct option is (b)**

51. An electron is moving with a speed of $0.85c$ in a direction opposite to that of a photon. The relative velocity of the electron w.r.t. the photon is **[BHU 2015]**

(a) $0.85c$ (b) $0.65c$ (c) $0.75c$ (d) $c$

**Soln.** $$v_{ep}=\frac{v_e+v_p}{1+\frac{v_e v_p}{c^2}}=\frac{0.85c+c}{1+\frac{0.85c^2}{c^2}}=\frac{1.85}{1.85}c=c.$$

**Correct option is (d)**

52. The energy equivalent of the rest mass of an electron is **[BHU 2015]**

(a) 1.12 MeV (b) 0.51 MeV (c) 0.91 MeV (d) 0.85 MeV

**Soln.** $$E=mc^2=(9.1\times10^{-31})\,(3\times10^{8})^2\ \text{J}$$
$$=\frac{(9.1\times10^{-31})\,(3\times10^{8})^2}{1.6\times10^{-19}}\ \text{eV}$$
$$=0.51\ \text{MeV}.$$

**Correct option is (b)**

53. The Lorentz invariant length between two space-time points $(x, y, z, ct)$ and $(x', y', z', ct')$ is as follows: **[BHU 2015]**

(a) $(x'-x)^2+(y'-y)^2+(z'-z)^2-c^2(t'-t)^2$
(b) $(x'-x)^2+(y'-y)^2-(z'-z)^2+c^2(t'-t)^2$
(c) $(x'-x)^2-(y'-y)^2+(z'-z)^2+c^2(t'-t)^2$
(d) $(x'-x)^2+(y'-y)^2+(z'-z)^2+c^2(t'-t)^2$

**Soln.** The Lorentz invariant length is $ds^2=(x'-x)^2+(y'-y)^2+(z'-z)^2-c^2(t'-t)^2$.

**Correct option is (a)**

54. Total energy of a relativistic particle, (which is the sum of kinetic energy and rest mass energy) is :

(a) $m_0c^2$ (b) $\dfrac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}$ (c) $m_0c^2\sqrt{1-\frac{v^2}{c^2}}$ (d) $\dfrac{m_0v^2}{\sqrt{1-\frac{v^2}{c^2}}}$ **[BHU 2015]**

where $m_0$ is the rest mass if the particle moving with velocity $v$ and $c$ is the speed of light.

**Soln.** The total energy of a relativistic particle is

$$E = mc^2 = \frac{m_0}{\sqrt{1-v^2/c^2}}c^2$$

**Correct option is (b)**

55. In the laboratory frame of reference, a particle moving with speed $2.8 \times 10^{10}$ cm/sec, has the life-time $2.5 \times 10^{-7}$ sec. The proper life-time of the particle is **[BHU 2015]**

(a) $\approx 5\times10^{-10}$ sec (b) $\approx 6\times10^{-11}$ sec (c) $\approx 7\times10^{-8}$ sec (d) $\approx 7\times10^{12}$ sec

**Soln.** The proper life-time is

$$\tau_0 = \tau\sqrt{1-v^2/c^2} = 2.5\times10^{-7}\sqrt{1-\left(\frac{2.8\times10^8}{3\times10^8}\right)^2} = 8.9\times10^{-8}\text{ sec}.$$

**Correct option is (b)**

56. The proper length of a rod is 100 cm. If the rod starts moving with speed 0.8*c*, its length would be

(a) 110 cm (b) 90 cm (c) 60 cm (d) 120 cm **[BHU 2015]**

**Soln.** If the proper length of a rod is $L_0$, then its length would be

$$L = L_0\sqrt{1-v^2/c^2} = 100\sqrt{1-(0.8)^2} = 60\text{ cm}.$$

**Correct option is (c)**

57. An example of a non-inertial reference frame is when **[BHU 2016]**

(a) a frame of reference is stationary w.r.t. the other frame

(b) a frame of reference moves with a uniform velocity towards the other reference frame

(c) a reference frame moves away from the other reference frame with a uniform velocity

(d) a reference frame rotates with a uniform angular velocity w.r.t. the other frame

**Soln.** An inertial reference frame is either at rest or moving with constant velocity with respect to another frame.

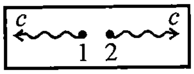
Therefore, reference frame mentioned option (d) is not an inertial frame.

**Correct option is (d)**

58. Two electrons move in the opposite directions each with speed 0.9 C. The relative velocity of one with respect to the other is **[BHU 2016]**

(a) 1.80 C (b) 0.99 C (c) 0.88 C (d) 0.77 C

**Soln.** Since, photons travel with the speed of light, so $v_1 = v_2 = c$.



Their relative speed is $v = \dfrac{v_1+v_2}{1+\dfrac{v_1v_2}{c^2}} = \dfrac{c+c}{1+\dfrac{c^2}{c^2}} = c$

**Correct option is (b)**

59. A clock moves away from an observer with a uniform velocity. This clock would seem to lose 1 minute per day if it moves with the velocity: **[BHU 2016]**

(a) $1.12\times10^7$ m/sec (b) $2.24\times10^7$ m/sec (c) $12.24\times10^7$ m/sec (d) $1.12\times10^6$ m/sec

**Soln.** If there are 1 day ≡ 1440 minutes on earth, there are 1339 minutes on the moving clock.

Using $t = \dfrac{t_0}{\sqrt{1-v^2/c^2}}$, we have

$$\Rightarrow \quad 1440 = \frac{1339}{\sqrt{1 - v^2/c^2}}$$

$$\text{or} \quad 1 - \frac{v^2}{c^2} = \left(\frac{1339}{1440}\right)^2$$

$$\text{or} \quad v = c\sqrt{1 - \left(\frac{1339}{1440}\right)^2} \simeq 1.103 \times 10^8 \text{ m/s}$$

$$= 11.03 \times 10^7 \text{ m/s}$$

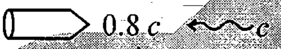
**Correct option is (c)**

60. An astronaut moves in a spaceship travelling at a speed of 0.8 *c* (*c* is the velocity of light), and observes a photon approaching him from space. The speed of photon with respect to the spaceship is
(a) 1.8 *c* (b) 0.2 *c* (c) 0.9 *c* (d) *c* **[HCU 2010]**

**Soln.** Given : $v_1 = 0.8c, \ v_2 = c$

The speed of the photon with respect to the spaceship is

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} = \frac{0.8c + c}{1 + \frac{0.8c^2}{c^2}} = \frac{1.8}{1.8}c = c.$$



**Correct option is (d)**

61. Special relativity predicts that an outside inertial observer sees a moving object as bearing one of the following relationships with respect to an observer moving with the object
(a) longer (b) shorter (c) brighter (d) unchanged **[HCU 2010]**

**Soln.** The outside inertial observer sees the object to be shorter than the moving observer with the object.

**Correct option is (b)**

62. The proper length of a space vehicle is $l_0$. According to an observer on earth, the length of the spaceship is 25% of its proper length. The speed of the spaceship according to the observer on earth is given by
(a) $\frac{\sqrt{3}}{2}c$ (b) $\sqrt{\frac{3}{2}}c$ (c) $c$ (d) $0.968\,c$ **[HCU 2010]**

**Soln.** Given : $L_0 = l_0$ and $L = \frac{l_0}{4}$

Using $L = L_0\sqrt{1 - v^2/c^2} \Rightarrow \frac{1}{4} = \frac{1 - v^2}{c^2} \Rightarrow v = \frac{\sqrt{3}}{2}c.$

**Correct option is (a)**

63. Two electrons move towards each other, the speed of each being 0.9 *c*. What is their relative velocity according to special theory of relativity ?
(a) 0.995 *c* (b) 1.09 *c* (c) 0.89 *c* (d) 0.91 *c* **[HCU 2010]**

**Soln.** The relative velocity is

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} = \frac{1.8c}{1.81} \simeq 0.995c. \qquad \underset{\bullet\!\longrightarrow}{v_1 = 0.9\,c} \qquad \underset{\longleftarrow\!\bullet}{v_2 = 0.9\,c}$$

**Correct option is (a)**

64. Two events occur at an interval $\Delta t$ at the same point in a rest frame. In a frame which is moving with a uniform velocity $v$, the time interval between the events will be given by

(a) $\frac{\Delta t}{\sqrt{1-v^2}}$ (b) $(\Delta t)\sqrt{\frac{1-v^2}{c^2}}$ (c) $\frac{\Delta t}{\sqrt{1-v^2/c^2}}$ (d) $\frac{\Delta t}{\sqrt{1+v^2/c^2}}$ **[HCU 2010]**

**Soln.** The two time instants are $t_1' = \frac{1}{\sqrt{1-v^2/c^2}}\left(t_1 - \frac{vx}{c^2}\right)$ and $t_2' = \frac{1}{\sqrt{1-v^2/c^2}}\left(t_2 - \frac{vx}{c^2}\right)$.

$$\therefore \ \Delta t' = \frac{1}{\sqrt{1-v^2/c^2}}(t_1 - t_2) = \frac{\Delta t}{\sqrt{1-v^2/c^2}}.$$

**Correct option is (c)**

65. A measuring rule of rest length $l$ moves relative to an observer with the velocity $v$. The observer measures the length of the rule as $\frac{2}{3}l$. What is the velocity $v$ ?

(a) $c$ (b) $0.1\,c$ (c) $0.745\,c$ (d) $0.345\,c$ **[HCU 2010]**

**Soln.** Given : $L_0 = l$ and $L = \frac{2}{3}l$

Using, $L = L_0\sqrt{1-v^2/c^2} \Rightarrow \frac{4}{9} = 1 - \frac{v^2}{c^2} \Rightarrow v = 0.745c.$

**Correct option is (c)**

66. An astronaut moves in a spaceship travelling at a speed of 0.8 c (c is the speed of light). If the astronaut observes a photon approaching the ship from space, then the speed of this photon with respect to the observer is

(a) 1.8 c (b) c (c) 0.2 c (d) 0.9 c **[HCU 2012]**

**Soln.** The speed of the photon with respect to the observer is

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} = \frac{0.8c + c}{1 + \frac{0.8c^2}{c^2}} = c.$$

**Correct option is (b)**

67. Two observes having identical instruments carry out identical experiments to study the motion of a massive particle. First observer concludes that the particle is moving in a straight line with constant velocity while the second observer concludes that the particle is moving with acceleration. Which of the following four statements made about these two observers are correct :

(i) both observers are at rest

(ii) second observer is moving with constant velocity with respect to the first observer

(iii) second observer is accelerating with respect to the first observer

(a) (i) (b) (ii) (c) (i) and (iii) (d) (iii) **[HCU 2014]**

**Soln.** If the particle is moving with a constant velocity w.r.t. an inertial frame, then it will also move with a constant velocity w.r.t. an another inertial frame.

Therefore, from the information given in the question we can say that both observers are not inertial i.e. second observer is accelerating w.r.t the first observer.

**Correct option is (d).**

68. A particle is accelerated such that its energy is three times the its rest mass energy. Then velocity of the particle would be

(a) $\sqrt{2}\dfrac{c}{3}$ (b) $\dfrac{3c}{2\sqrt{2}}$ (c) $2\sqrt{2}\dfrac{c}{3}$ (d) $\dfrac{c}{\sqrt{3}}$ **[HCU 2014]**

**Soln.** The energy is given by $E = mc^2$

$$\Rightarrow 3m_0c^2 = \frac{m_0}{\sqrt{1-v^2/c^2}}c^2 \Rightarrow 1-\frac{v^2}{c^2}=\frac{1}{9} \text{ or } v=\frac{2\sqrt{2}}{3}c.$$

**Correct option is (c)**

69. A particle of mass $1.40\times10^5$ kg has a relativistic momentum of magnitude $3.15\times10^{13}$ kg m/s. How fast is the particle traveling? **[HCU 2015]**

(a) $2.68\times10^8\ m/s$ (b) $1.8\times10^8\ m/s$ (c) $3.3\times10^{16}\ m/s$ (d) $3\times10^8\ m/s$

**Soln.** The relativistic moment of a particle is given by $p=\dfrac{m_0}{\sqrt{1-v^2/c^2}}v$.

Given: $p = 3.15\times10^{13}$ kg-m/sec, $m_0 = 1.40\times10^5$ kg

$$\therefore\ 2.25\times10^8 = \frac{v}{\sqrt{1-v^2/c^2}} \Rightarrow (2.25\times10^8)^2 = \frac{v^2}{1-v^2/c^2}$$

$$\Rightarrow v = \sqrt{\frac{(2.25\times10^8)^2}{1+\dfrac{(2.25\times10^8)}{3\times10^8}}} = 1.8\times10^8 \text{ m/sec.}$$

**Correct option is (b)**

70. A proton of rest mass $1.68\times10^{-27}\ kg$ is traveling at $2.5\times10^8\ m/s$. What is kinetic energy of the proton? **[HCU 2015]**

(a) $1.18\times10^{-10}\ J$ (b) $0.18\times10^{-10}\ J$ (c) $2.18\times10^{-27}\ J$ (d) $1.32\times10^{-27}\ J$

**Soln.** The kinetic energy of the proton is given by

$$K = mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} - m_0c^2.$$

Given: $m_0 = 1.68\times10^{-27}$ kg, $v = 2.5\times10^8$ m/sec.

$$\therefore\ K = \frac{1.68\times10^{-27}\times9\times10^6}{\sqrt{1-\left(\dfrac{2.5}{3}\right)^2}} - 1.68\times10^{-27}\times9\times10^{16}$$

$$= \left(\frac{15.12\times10^{-11}}{0.552} - 15.12\times10^{-11}\right)\text{J} \quad = (27.35\times10^{-11} - 15.12\times10^{-11})\text{J} \quad = 1.22\times10^{-10}\text{ J}$$

**Correct option is (a)**

71. A spaceship passes you at a speed of 0.85 c. You meausre its length to be 48.2 m. What is its length at rest?

(a) 80 m (b) 100 m (c) 91.5 m (d) 81 m **[HCU 2015]**

**Soln.** According to Lorentz transform, $L_0' = L_0\sqrt{1-v^2/c^2}$

$$\Rightarrow 48.2 = L_0\sqrt{1-(0.85)^2} \Rightarrow L_0 = 91.49\text{ m} \simeq 91.5\text{ m}.$$

**Correct option is (c)**

72. A particle at rest has a lifetime of 3 μsec. An observer measures the speed of this particle to be $\frac{3c}{5}$. The distance travelled by this particle, measured by the observer, is **[HCU 2016]**

(a) $\frac{9c}{4}\times10^{-6}\ m$ (b) $\frac{15c}{4}\times10^{-6}\ m$ (c) $\frac{18c}{5}\times10^{-6}\ m$ (d) $\frac{9c}{16}\times10^{-6}\ m$

**Soln.** The life time of the particle as measured by the observer in its own frame is

$$\tau = \frac{\tau_0}{\sqrt{1-v^2/c^2}} = \frac{3\mu\sec}{\sqrt{1-\left(\frac{3c}{5c}\right)^2}} = \frac{15}{4}\mu\sec.$$

Therefore, distance travelled = $v\tau$

$$= \frac{3c}{5}\times\frac{15}{4}\mu\sec = \frac{9c}{4}\times10^{-6}\text{ m}.$$

**Correct option is (a)**

74. Two electrons are ejected in opposite directions from a sample of radioactive atoms. Each electron has a speed of 0.67c as measured by a laboratory observer. Their relative velocity is given by **[HCU 2013]**

(a) 1.34 c (b) 1.19 c (c) 0.92 c (d) 0.87 c

**Soln.** The velocity of the pairs of electron w.r.t. lab frame are $v_1 = 0.67c$ and $v_2 = 0.67c$.

The velocity of electron w.r.t. other electron is

$$v = \frac{v_1+v_2}{1+\frac{v_1v_2}{c^2}}$$

$$= \frac{0.67c+0.67c}{1+\frac{0.67c\times0.67c}{c^2}} = 0.92c.$$

**Correct option is (c)**

75. A muon, whose life-time at rest is $2\times10^{-6}$ sec, is travelling through the laboratory at three-fifth of the speed of light. It will last in **[DU 2014]**

(a) $2\times10^{-6}$ sec. (b) $2.5\times10^{-6}$ sec (c) $3.5\times10^{-6}$ sec (d) $3.0\times10^{-5}$ sec

**Soln.** Life-time of the muon w.r.t Laboratoty will be

$$t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2\times10^{-6}}{\sqrt{1-(0.6)^2}}s = 2.5\times10^{-6}s$$

**Correct option is (b)**

76. Relative to a stationary observer, a rod of length 1.0 metre is moving at 0.8 times the speed of light in vacuum. It would appear to the observer that the rod's length is: **[DU 2015]**

(a) 0.8 m (b) 0.6 m (c) 1.0 m (d) 1.25 m

**Soln.** Length of the rod w.r.t. the observer will be

$$l = l_0\sqrt{1-\frac{v^2}{c^2}} = 1\times\sqrt{1-(0.8)^2}\,m = 0.6\,m$$

**Correct option is (b)**

77. Two events take place at the same place in a lab frame but occur with a time difference of 3 seconds. The same events occur with a time difference of 5 seconds in a rocket frame. The relative speed of rocket and the laboratory is **[DU 2016]**

(a) 0.5c (b) 0.8c

(c) 0.6c (d) cannot be determined from this data

**Soln.** The relation between $\Delta t'$ (the time in the rocket frame) and $\Delta t$ (the time in the lab frame) if the two events take place at the same location in a lab frame is $\Delta t' = \frac{\Delta t}{\sqrt{1-v^2/c^2}}$.

$$\Rightarrow 1-\frac{v^2}{c^2} = \left(\frac{\Delta t}{\Delta t'}\right)^2 = \left(\frac{3}{5}\right)^2 \Rightarrow v = \frac{4}{5}c = 0.8c.$$

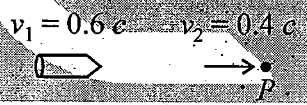
**Correct option is (b)**

78. A spaceship approaches a planet with a speed 0.6c. At some point it fires a projectile with a speed 0.4c towards the planet. The speed of the projectile, as seen by an observer on the planet would be approximately **[DU 2016]**

(a) 0.76c (b) 0.80c (c) 0.99c (d) 0.40c

**Soln.** The speed of the projectile as seen by an observer on the planet is

$$v = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}} = \frac{0.6c+0.4c}{1+\frac{0.24c^2}{c^2}} = \frac{1}{1.24}c = 0.80c.$$



**Correct option is (b)**

79. A particle is moving at a speed of $2.6\times10^8\ ms^{-1}$ relative to the laboratory. Its lifetime as measured by an observer in the laboratory is $4.7\times10^{-6}$ s. The lifetime of the particle in its own rest frame is about **[DU 2016]**

(a) $2.3\times10^{-6}s$ (b) $9.4\times10^{-6}s$ (c) $4.7\times10^{-6}s$ (d) $14.4\times10^{2}s$

**Soln.** Given : $\tau = 4.7\times10^{-6}$ sec, $v = 2.6\times10^{-8}$ m/s.

The lifetime of the particle in its own frame is

$$\tau_0 = \tau\sqrt{1-v^2/c^2}$$

$$= 4.7\times10^{-6}\sqrt{1-\left(\frac{2.6\times10^8}{3\times10^8}\right)^2}$$

$$= 2.3\times10^{-6}\ \text{sec}.$$

**Correct option is (a)**

# Chapter 10

# Atomic Physics

## 10.1. Bohr's Atom Model :

**Bohr's postulates :**

The postulates of Bohr's theory for hydrogen atom are enumerated.

(1) An electron in atom moves in a circular orbit about the nucleus under the influence of coulomb attraction between the electron and the nucleus, and obeys the laws of classical mechanics.

(2) But instead of an infinite number of orbits which would be possible in classical mechanics, there are only certain discrete orbits (quantization) for which the angular momentum of the electron, L, is an integral multiple of

$h/2\pi\left(=\hbar\right)$, where $h$ is Planck's constant.

i.e. $L = nh/2\pi = n\hbar$; $n = 1, 2, \ldots$etc.

(3) An electron moving in such a quantised orbit (also called stationary energy state) does not radiate electromagnetic energy, as predicted by electromagnetic theory. Thus, its total energy remains constant.

(4) Electromagnetic radiation is either emitted or absorbed only when transition of an electron between two stationary states takes place. The radiation is emitted or absorbed as a single quantum whose energy E is equal to the difference in energy of the two states, and is given by E = h$\nu$, according to Planck's law where '$\nu$' is the frequency of the radiation.

$\therefore$ $E = E_m - E_n = h\nu_{nm}$ (Bohr's frequency condition)

**Note :** If the electron is initially at the lower state $E_1$ then transition to the higher state $E_2$ is possible only if it absorbs an amount of energy $E = E_2 - E_1 = h\nu$ incident on it.

**Expression for orbital radius:**

Consider a hydrogen-like atom of atomic number Z consisting of a nucleus of charge +Ze (Z = 1 for hydrogen) and a single electron of mass $m$ and charge $-e$ revolving in a circular orbit of radius $r_n$. For the mechanical stability of the electron,

$$\frac{Ze^2}{4\pi\varepsilon_0 r_n^2} = \frac{mv_n^2}{r_n} \qquad \ldots \text{(i)}$$

where, $v_n$ is the velocity of the electron in its orbit of radius $r_n$. The left side of (i) is the Coulomb pull that provides the required centripetal force $\dfrac{mv^2}{r}$ to keep the electron in orbit.

From Bohr's quantum condition, we obtain

$$L = I\omega = mr_n^2\left(\frac{v_n}{r_n}\right) = mv_n r_n = \frac{nh}{2\pi} \qquad \text{... (ii)}$$

From (i) and (ii), eliminating $r_n$, we get

$$Ze^2 = 4\pi \in_0 v_n (nh/2\pi)$$

$$\therefore \qquad v_n = \frac{Ze^2}{2 \in_0 nh} \qquad \text{... (iii)}$$

From (ii) and (iii),

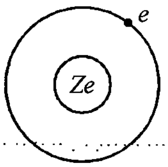
$$r_n = \frac{n^2 h^2 \in_0}{\pi m Z e^2} \qquad \text{... (iv)}$$

The radii $r_n$ and the electron's orbit velocity $v_n$ both depend on the integer $n$, and the radii is proportional to the square of the $n$-value, $n$ being known as a quantum number, specifically the principal quantum number.

**Expression for total energy:**

The total energy $E_n$ of the electron in its n<sup>th</sup> orbit is the sum of its kinetic and potential energy.

Kinetic energy, $E_k = \frac{1}{2}mv_n^2 = Ze^2/8\pi \in_0 r_n$, using the relation (i)



Potential energy, $V = \int_{\infty}^{r_n} \frac{Ze^2}{4\pi \in_0 r_n^2}\, dr_n = -\frac{Ze^2}{4\pi \in_0 r_n}$

Therefore, total energy, $E_n = E_k + V = \frac{Ze^2}{8\pi \in_0 r_n} - \frac{Ze^2}{4\pi \in_0 r_n} = -\frac{Ze^2}{8\pi \in_0 r_n}$ ... (v)

Therefore, energy in the nth orbit, $E_n = -\frac{Ze^2}{8\pi \in_0}\left(\frac{\pi m Z e^2}{n^2 h^2 \in_0}\right)$, using the relation (iv)

$$\text{Or,} \qquad E_n = -\left(\frac{1}{n^2}\right)\frac{mZ^2 e^4}{8 \in_0^2 h^2} \qquad \text{... (vi)}$$

The energy of electron within an atom is also quantised, that is, it can take only certain discrete values and comes out to be negative. So, as $n$ increases, i.e., as electron goes to orbits of larger radii, $E_n$ increases and converges to the limiting value $E_n = 0$ as $n \to \infty$. $E_n$ is lowest for the smallest orbit for which $n = 1$.

**Frequency of emitted radiation:**

Let the electron make a transition from the quantum state $n_i$ (corresponding to total energy $E_i$) to the quantum state $n_f$ (corresponding to total energy $E_f$). If $n_i > n_f, E_i > E_f$, then there will be emission of radiation of frequency $v$, given by Bohr's frequency condition,

$$v = \frac{E_i - E_f}{h} = \frac{mZ^2e^4}{8\in_0^2 h^3}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \qquad \ldots \text{(vii)}$$

In terms of wave number, $\bar{v}$ (defined as reciprocal of wavelength, that is, $\bar{v} = \frac{1}{\lambda} = \frac{v}{c}$)

$$\bar{v} = \frac{me^4}{8\in_0^2 ch^3} Z^2\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) = R_\infty Z^2\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \qquad \ldots \text{(viii)}$$

where, $R_\infty = \frac{me^4}{8\in_0^2 ch^3}$, a constant whose value is $1.09737 \times 10^7\ m^{-1}$

**Origin of spectral series:**

The orgin of spectral series can easily be understood now from Bohr's theory.

By putting $n_f = 2$ and $n_i = 3, 4, 5, \ldots\ldots$etc in the equation (viii), we get a series of spectral lines of wave number.

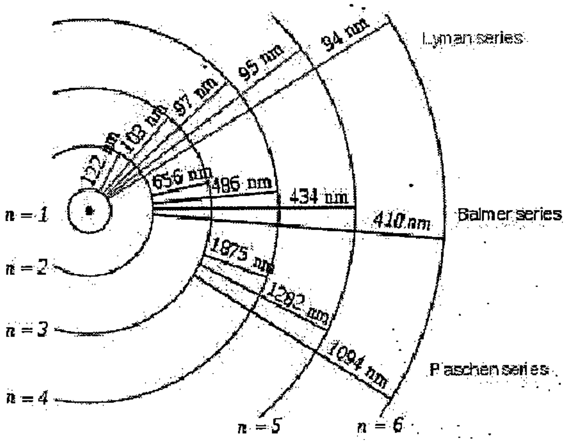
$$\bar{v} = R_\infty Z^2\left(\frac{1}{2^2} - \frac{1}{n_i^2}\right), \quad n_i = 3, 4, \ldots\ldots etc.$$

$$= R_\infty\left(\frac{1}{2^2} - \frac{1}{n_i^2}\right) \quad \text{for hydrogen (Z = 1)}$$

The series correponds to the Balmer series of hydrogen as given by the empirical formula.

$$\bar{v} = R_H\left(\frac{1}{2^2} - \frac{1}{n_i^2}\right) \qquad \text{where, } R_H = R_\infty = \frac{me^4}{8\in_0^2 ch^3}$$

The constant $R_H$ is known as the Rydberg constant.


94 nm
95 nm
97 nm
103 nm
122 nm
Lyman series
656 nm
486 nm
434 nm
410 nm
Balmer series
1875 nm
1282 nm
1094 nm
Paschen series
n = 1
n = 2
n = 3
n = 4
n = 5
n = 6

The equation (viii) suggests that series other than Balmer should also exist, corresponding to $n_f = 1, n_f = 3, n_f = 5$ etc. in regions of the spectrum away from the visible. Thus, the other series lines emitted by hydrogen are :

**Lyman series,** $\bar{\nu} = R_\infty\left(\frac{1}{1^2} - \frac{1}{n^2}\right)$, $n = 2, 3, 4, \ldots\ldots\ldots$

**Balmer series,** $\bar{\nu} = R_\infty\left(\frac{1}{2^2} - \frac{1}{n^2}\right)$, $n = 3, 4, 5, \ldots\ldots\ldots$

**Paschen series,** $\bar{\nu} = R_\infty\left(\frac{1}{3^2} - \frac{1}{n^2}\right)$, $n = 4, 5, 6, \ldots\ldots\ldots$

**Brackett series,** $\bar{\nu} = R_\infty\left(\frac{1}{4^2} - \frac{1}{n^2}\right)$, $n = 5, 6, 7, \ldots\ldots\ldots$

**Pfund series,** $\bar{\nu} = R_\infty\left(\frac{1}{5^2} - \frac{1}{n^2}\right)$, $n = 6, 7, 8, \ldots\ldots\ldots$

### Correction in Bohr's theory for finite nuclear mass

In the general Bohr's theory, the nucleus is considered stationary, that is, it is assumed to be infinitely massive compared to the mass of the electron. However, a nucleus of finite mass cannot be at rest. To modify the theory on this account is rather simple.

Consider the electron-nucleus system as shown in figure. Both the electron of mass $m$ and the nucleus of mass $M$ must move such that the centre-of-mass $O$ of the composite system remains fixed in space.

Let $r_1$ and $r_2$ be the radii of the orbits (assumed circular) of the electron and the nucleus respectively. Then,

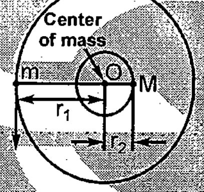


Fig. Correction for finite nucleus mass

$$mr_1 = Mr_2 \Rightarrow r_2 = \frac{m}{M} r_1 \qquad \ldots(i)$$

Therefore, the distance the nucleus and the electron

$$r = r_1 + r_2 = r_1\left(1 + \frac{m}{M}\right), \text{ using (i)} \qquad \ldots(ii)$$

The electron and the nucleus must rotate about the centre-of-mass with a common angular velocity $\omega$. So the total angular momentum $L$ of the system is given by

$$L = mr_1^2\omega + Mr_2^2\omega$$

$$= mr_1^2\omega\left(1 + \frac{m}{M}\right), \text{ using (i)}$$

$$= \frac{mr^2\omega}{1 + \frac{m}{M}}, \text{ using (ii)}$$

$$= \frac{mM}{m + M} rv \quad (\because\ v = r\omega) \qquad \ldots(iii)$$

For an infinitely heavy nucleus, the electron rotates about the stationary nucleus with

$$L = mrv \qquad \ldots(iv)$$

Thus the net effect is to replace the mass $m$ of the electron by the so called reduced mass

$$\frac{1}{\mu}=\frac{1}{M}+\frac{1}{m} \qquad \text{...(v)}$$

The Rydberg constant would now become

$$R_H=\frac{\mu e^4}{8\varepsilon_0^2 ch^3}=\frac{m}{\left(1+\dfrac{m}{M}\right)}\left(\frac{e^4}{8\varepsilon_0^2 ch^3}\right)=\frac{R_\infty}{1+\dfrac{m}{M}}$$

where $R_\infty$ is the Rydberg constant for an infinitely heavy mass.

It is clear from (v) that the reduced mass $\mu < m$ and increases with increasing nucleus mass $M$. When $M \to \infty$, $\mu \to m$, the electronic mass. Therefore, $R_H$ increases as $M$ increases and as $M \to \infty$, $R_M \to R_\infty$. It is observed that the spectral lines of Pickering series of He$^+$ have slightly higher wave numbers of shorter wavelengths than the corresponding lines of Balmer series of hydrogen. The explanation lies in the fact that $R_{\text{He}} > R_H$.

## Bohr's correspondence principle

Neither in Bohr's theory nor in Planck's quantum which Bohr utilised, any attempt was made in developing any new theory of atomic mechanics. Hence they were *ad hoc* in nature. Bohr however noted from the energy expression that as the quantum number $n$ becomes very large, the successive values of the allowed energy differ very little from each other and appear almost continuous. Also, as the spectral lines tend to the series limit they crowd so much that the spectrum appears continuous. In this context, Bohr in 1923 enunciated an important principle called the correspondence principle. It states that the behaviour of an atomic system as predicted by quantum theory tends asymptotically (i.e. corresponds) to that expected in classical theory in transitions involving states of large quantum numbers. Bohr's correspondence principle forms an important guideline in formulating the laws of atomic mechanics.

We give here an elementary deduction (illustration) of the principle.

The frequency of revolution $f$ of the electron in an orbit of radius $r$ is

$$f=\frac{\text{electron speed}}{\text{orbit circumference}}=\frac{v}{2\pi r}$$

Substituting the values of $v$ and $r$

$$v_n=\frac{Ze^2}{2\in_0 nh} \; ; \; r=\frac{n^2h^2\in_0}{\pi mZe^2}$$

$$f=\frac{mZ^2e^4}{8\varepsilon_0^2h^3}\left(\frac{2}{n^3}\right) \qquad \text{...(i)}$$

A radiated frequency in quantum theory however is, from (vii) of Art 1.2,

$$\nu=\frac{mZ^2e^4}{8\varepsilon_0^2h^3}\left(\frac{1}{n_f^2}-\frac{1}{n_i^2}\right)=\frac{mZ^2e^4}{8\varepsilon_0^2h^3}\,\frac{(n_i-n_f)(n_i+n_f)}{n_i^2\,n_f^2}$$

$$=\frac{mZ^2e^4}{8\varepsilon_0^2h^3}\,\frac{(2np-p^2)}{n^2(n-p)^2} \qquad \text{...(ii)}$$

writing $n$ for $n_i$ and $n-p\ (p=1,2,3,\ldots)$ for $n_f$.

For very large $n_i$ and $n_f$, $n \gg p$ and we obtain

$$\frac{2np-p^2}{n^2(n-p)^2}\simeq\frac{2np}{n^4}$$

Therefore, from (ii) above, $v = \dfrac{mZ^2e^4}{8\varepsilon_0^2h^3}\left(\dfrac{2p}{n^3}\right)$.

When $p = 1, v = f$, the frequency of rotation of electron, from (i). Harmonies of this frequency are radiated when $p = 2, 3, 4, \ldots$. Thus for very large quantum numbers, the classical orbit frequency equals the quantum theory frequency.

**Example 1.** Calculate the velocity, radius and energy of the first Bohr orbit and also the Rydberg constant for *H*-atom.

**Soln.** **Orbital speed :** We have from the relation, $v_n = \dfrac{e^2}{2h\varepsilon_0}\left(\dfrac{Z}{n}\right)$

Therefore, $v_1 = \dfrac{e^2}{2h\varepsilon_0} (\because\ Z = 1, n = 1) = \dfrac{(1.6\times10^{-19})^2}{2\times6.62\times10^{-34}\times8.85\times10^{-12}} = 2.18\times10^6$ m/s.

**Orbital radius :** $r_n = \dfrac{n^2h^2\varepsilon_0}{\pi me^2} (\because\ Z = 1)$

Therefore, $r_1 = \dfrac{h^2\varepsilon_0}{\pi me^2} (\because\ n = 1) = \dfrac{(6.62\times10^{-34})^2\times8.85\times10^{-12}}{3.14\times9.1\times10^{-31}\times(1.6\times10^{-19})^2} = 0.53\times10^{-10}\,\text{m} = 0.53\ \text{Å}$

**Orbital energy :** $E_1 = \dfrac{me^4}{8\varepsilon_0^2h^2}$ (save the sign)

Therefore, $E_1 = \dfrac{9.1\times10^{-31}\times(1.6\times10^{-19})^4}{8\times(8.85\times10^{-12})^2\times(6.62\times10^{-34})^2} = 2.17\times10^{-18}\,\text{J} = 13.6\,\text{eV}$

**Rydberg constant :** $R_H = \dfrac{me^4}{8\varepsilon_0^2ch^3}$

Therefore, $R_H = \dfrac{9.1\times10^{-31}\times(1.6\times10^{-19})^4}{8\times(8.85\times10^{-12})^2\times3\times10^8\times(6.62\times10^{-34})^3} = 1.093\times10^7\ \text{m}^{-1}$

**Note:** The radius of the first Bohr orbit of hydrogen is called Bohr radius.

**Example 2.** Show that the energy required to raise *H*-atom from the ground state ($n = 1$) to the first excited state ($n = 2$) is about 10 eV.

**Soln.** Let $E_1$, $E_2$ be the energies of the electron in states $n = 1$ and $n = 2$ respectively.

Therefore, $E_1 = \dfrac{me^4}{8\varepsilon_0^2h^2}$; $E_2 = \dfrac{me^4}{2^2\times8\varepsilon_0^2h^2}$

Therefore, required energy $E = E_1 - E_2 = \dfrac{3me^4}{4\times8\varepsilon_0^2h^2}$

Therefore, $E = \dfrac{3\times9.1\times10^{-31}\times(1.60\times10^{-19})^4}{32\times(8.85\times10^{-12})^2\times(6.62\times10^{-34})^2}\ \text{J} \simeq 10\,\text{eV}$

**Example 3.** Given the Rydberg constant as $1.097 \times 10^7\ \text{m}^{-1}$, calculate the wavelength of the first line of Balmer series in Angstrom unit.

**Soln.** For the first line i.e. $H_\alpha$-line of Balmer series we have $n_i = 3, n_f = 2$.

Therefore, Wave number of $H_\alpha$-line : $\bar{v} = R_H\left(\frac{1}{2^2} - \frac{1}{3^2}\right) = \frac{5}{36}R_H$

Therefore, Wavelength of $H_\alpha$-line : $\lambda = \frac{1}{\bar{v}} = \frac{36}{5R_H} = \frac{36}{5 \times 1.097 \times 10^7}\ \text{m} = 6563\ \text{A}$

**Example 4.** What is the velocity of electron in the ground state of Bohr's H-atom in terms of the speed of light? What is this called?

**Soln.** From the relation $v_n = \frac{e^2 Z}{2\varepsilon_0 hn}$, we have $v_1 = \frac{e^2}{2\varepsilon_0 h}$ $(\because\ Z = 1, n = 1)$. In terms of the speed of light, the electron velocity is $\frac{e^2}{2\varepsilon_0 ch} = \frac{(1.60 \times 10^{-19})^2}{2 \times 8.85 \times 10^{-12} \times 3 \times 10^8 \times 6.62 \times 10^{-34}} \simeq \frac{1}{137}$

This known as the fine structure constant.

**Example 5.** Calculate what will be the appropriate quantum number $n$ for an electron in an orbit of radius 0.1 mm.

**Soln.** The radius of Bohr's orbit, in terms of Bohr's radius $r_1$, is given by $r_n = 0.53n^2\ \text{Å}$.

For $r_n = 0.1\ \text{mm} = 10^6\ \text{Å}$, we therefore have from above $10^6 = 0.53n^2$

$\therefore\ n^2 = \frac{10^6}{0.53} = 18.87 \times 10^5$

$\therefore\ n \simeq 1374$

**Example 6.** The average of life time of an electron in an excited state of H-atom is about $10^{-8}$ s. How many revolutions does an electron in the $n = 2$ state make before its transition to $n = 1$ state. The Rydberg constant for H-atom is $1.097 \times 10^7\ \text{m}^{-1}$.

**Soln.** The frequency $f$ of revolution of the electron in an orbit is $f = \frac{\text{electron speed}}{\text{orbital circumference}} = \frac{v_n}{2\pi r_n}$.

But $v_n = \frac{Ze^2}{2\varepsilon_0 nh}$ and $r_n = \frac{n^2 h^2 \varepsilon_0}{\pi m Z e^2}$. $\therefore\ f = \frac{mZ^2 e^4}{8\varepsilon_0^2 h^3}\left(\frac{2}{n^3}\right) = \frac{2R_{H^c}}{n^3}$.

For $n = 2,\ f = \frac{2R_{H^c}}{8} = \frac{2 \times 1.097 \times 10^7 \times 3 \times 10^8}{8} = 8.2 \times 10^{14}\ \text{s}^{-1}$

Therefore, number of revolution in $10^{-8}$ s is $N = f \times 10^{-8} = 8.2 \times 10^{14} \times 10^{-8} = 8.2 \times 10^6$.

**Example 7.** Calculate the series limit of the Balmer series given $R_H = 1.097 \times 10^7\ \text{m}^{-1}$

**Soln.** The wavelength of spectral lines of the Balmer series are given by

$$\frac{1}{\lambda} = R_H\left(\frac{1}{2^2} - \frac{1}{n^2}\right);\ n = 3, 4, 5, \ldots.$$

For the series limit, $n = \infty$. We thus obtain

$$\frac{1}{\lambda}=R_H\left(\frac{1}{2^2}-\frac{1}{\infty^2}\right)=\frac{R_H}{4}$$

$$\therefore\ \lambda_\infty=\frac{4}{R_H}=\frac{4}{1.097\times10^7}=3.646\times10^{-7}\,\text{m}=3646\,\text{Å}$$

**Example 8.** The first line of the Balmer series of hydrogen has a wavelength 6563 Å. Calculate the wavelength of the second line.

**Soln.** The wavelengths of the spectral lines of Balmer series are given by $\frac{1}{\lambda}=R_H\left(\frac{1}{2^2}-\frac{1}{n^2}\right)$; $n=3,4,5,\ldots.$

For the first line, $n=3$ so that the wavelength $\lambda_1$ is given by $\frac{1}{\lambda_1}=R_H\left(\frac{1}{2^2}-\frac{1}{3^2}\right)=\frac{5R_H}{36}$.

For the second line $n=4$. Therefore, the wavelength $\lambda_2$ is given by $\frac{1}{\lambda_2}=R_H\left(\frac{1}{2^2}-\frac{1}{4^2}\right)=\frac{3R_H}{16}$

$$\therefore\ \frac{\lambda_1}{\lambda_2}=\frac{3R_H}{16}\times\frac{36}{5R_H}=\frac{27}{20}$$

$$\therefore\ \lambda_2=\lambda_1\times\frac{20}{27}=6363\times\frac{20}{27}=4861\,\text{Å}.$$

**Example 9.** Calculate the ionisation potential of (i) H-atom and (ii) He-atom in the ground state.

**Soln.** We have : $E_n=-\frac{me^4Z^2}{8\varepsilon_0^2n^2h^2}$

(i) For H-atom in ground state $Z=1, n=1$.

$$\therefore\ E_1=-\frac{9.1\times10^{-31}\times(1.60\times10^{-19})^4}{8\times(8.85\times10^{-12})^2\times(6.62\times10^{-34})^2}\,\text{J}=-13.6\,\text{eV}.$$

$\therefore$ Energy of the first Bohr orbit of H-atom = −13.6 eV

$\therefore$ Ionisation potential of H-atom in ground state = 13.6 eV

(ii) For the He-atom $Z=2$.

$$\therefore\ (E_1)_{He}=-\frac{me^4}{8\varepsilon_0^2h^2}\times4=+4(E_1)_H$$

$$\therefore\ (E_1)_{He}=+4\times(-13.6)=-54.4\,\text{eV}$$

$\therefore$ Ionisation potential of He-atom = 54.4 eV

**Example 10.** Find the wavelength separation of the first member of the Balmer series due to $^1$H and $^2$H(= $^2$D).

**Soln.** We have $\bar{\nu}_H=R_H\left(\frac{1}{2^2}-\frac{1}{n^2}\right)$.... for $^1$H and $\bar{\nu}_D=R_D\left(\frac{1}{2^2}-\frac{1}{n^2}\right)$.... for $^2$H(= $^2$D).

$$\therefore\ \frac{\bar{\nu}_H}{\bar{\nu}_D}=\frac{R_H}{R_D}=\frac{1+m/M_D}{1+m/M_H}=\frac{1+m/2M_H}{1+m/M_H}\ \text{or}\ \frac{\bar{\nu}_H\sim\bar{\nu}_D}{\bar{\nu}_D}=\frac{1}{2(1+M_H/m)}=\frac{1}{2\times1840}=\frac{1}{3680}$$

$$\therefore\quad \frac{\Delta\bar{v}}{\bar{v}}=\frac{\Delta\lambda}{\lambda}=\frac{1}{3680}$$

At $\lambda = 656.3$ nm, $\Delta\lambda = \lambda/3680 = 656.3/3680 = 0.178$ nm.

**Example 11.** A positronium atom is a system that consists of a positron and an electron. Calculate the reduced mass, the Rydberg constant and the wavelength of the first Balmer line for positronium. Given $m = 9.1\times10^{-31}\,kg$, $R_H = 1.09737\times10^{-3}\,\text{Å}^{-1}$ and $A_\alpha = 6563\,\text{Å}$.

**Soln.** The positron has the same mass $m$ as the electron and has equal but positive charge

$\therefore$ Reduced mass of positronium is

$$\mu=\frac{(m)(m)}{m+m}=\frac{1}{2}m=\frac{1}{2}\times9.1\times10^{-31}\,\text{kg}$$

$$=4.55\times10^{-31}\,\text{kg}$$

The Rydberg constant $\left(=\dfrac{\mu e^4}{8\varepsilon_0^2ch^3}\right)$ for positronium is therefore half that for hydrogen.

$$\therefore\quad R_P=\frac{1}{2}R_H=\frac{1}{2}\times1.09737\times10^{-3}\,\text{Å}^{-1}$$

$$=0.54868\times10^{-3}\,\text{Å}^{-1}$$

The wavelength of first Balmer line, $H_\alpha$, in $H$-atom is given by $\dfrac{1}{\lambda_H}=R_H\left(\dfrac{1}{2^2}-\dfrac{1}{3^2}\right)$, while that in positronium is given by $\dfrac{1}{\lambda_P}=R_P\left(\dfrac{1}{2^2}-\dfrac{1}{3^2}\right)$.

$$\therefore\quad \frac{\lambda_P}{\lambda_H}=\frac{R_H}{R_P}=2$$

$$\therefore\quad \lambda_P=2\lambda_H=2\times6563\,\text{Å}=13126\,\text{Å}$$

**Example 12.** An electron circles a nucleus of charge $Ze$. Of the two orbits 1 and 2 of radii $r_1$ and $r_2$ respectively, its total energy is greater while in orbit 1. Prove that $r_1 > r_2$, also show that the velocity and acceleration in orbit 2 are greater than those in orbit 1.

**Soln.** Energy of electron, $E=-\dfrac{Ze^2}{8\pi\varepsilon_0 r}=-\dfrac{1}{2}mv^2$ ...(i)

Since $E_1 > E_2$, we have $E_1 - E_2 > 0$ or $-\dfrac{Ze^2}{8\pi\varepsilon_0 r_1}+\dfrac{Ze^2}{8\pi\varepsilon_0 r_2}>0$, using (i)

or $\dfrac{Ze^2}{8\pi\varepsilon_0}\left(\dfrac{1}{r_2}-\dfrac{1}{r_1}\right)>0$ ...(ii)

$\therefore\quad r_1 > r_2$

From equation (i), again, $\dfrac{1}{2}m\left(v_2^2-v_1^2\right)=\dfrac{Ze^2}{8\pi\varepsilon_0}\left(\dfrac{1}{r_2}-\dfrac{1}{r_1}\right)$ or $v_2^2-v_1^2>0$, using (ii) above

$\therefore \; v_2 > v_1$

The acceleration of electron in circular orbit is the normal acceleration, $f = v^2/r$.

We have : $\dfrac{mv^2}{r} = \dfrac{Ze^2}{4\pi\varepsilon_0 r^2}$

$$\therefore \; f_n = \frac{Ze^2}{4\pi m\varepsilon_0 r_n^2}$$

$$\therefore \; f_2 - f_1 = \frac{Ze^2}{4\pi m\varepsilon_0}\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right) \text{ or } f_2 - f_1 > 0 \; (\because r_1 > r_2)$$

$\therefore \; f_2 > f_1$

**Example 13.** Show that in a Bohr atom if the electron is considered as a wave travelling along the circular path, then the orbit will contain $n$ complete de-Broglie waves.

**Soln.** The radius of the $n$th Bohr orbit is given by $r_n = \dfrac{n^2h^2\varepsilon_0}{\pi me^2}$

Therefore, circumference of the orbit is $2\pi r_n = \dfrac{2n^2h^2\varepsilon_0}{me^2}$. But the wavelength of the de-Broglie wave is given by $\lambda = \dfrac{h}{mv} = \dfrac{h}{m}\left(\dfrac{2nh\varepsilon_0}{e^2}\right)\left(\because v_n = \dfrac{e^2}{2nh\varepsilon_0}\right)$.

Therefore, number of complete de-Broglie waves in the $n$th orbit is $\dfrac{2n^2h^2\varepsilon_0}{me^2} \div \dfrac{h}{m}\left(\dfrac{2nh\varepsilon_0}{e^2}\right) = n$.

## 10.2. Stern-Gerlach Experiment:

Stern and Gerlach, in 1921, perfomed an experiment which demonstrated directly that an atom placed in a magnetic field can take only certain discrete orientations with respect to the field (space quantisation). It also demonstrated the existence of electron spin, thus providing an experimental verification of the vector model of atom.

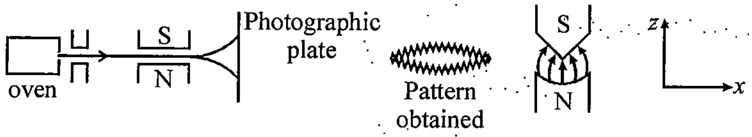
The plan of the experiment is shown in figure (below). A beam of neutral silver atoms was formed by heating silver in an oven. It was collimated by a few fine slits and then passed through a non-homogeneous magnetic field. The beam leaving the mgnetic field was received on a photographic plate P.

**Physical Significance:**

(i) Experimental evidence of electron spin.

(ii) Direct demonstration of space quantisation.

**Experiment :** The experiment of Stern and Gerlach was intended to study the deflection of atomic beams in non-uniform magnetic fields.



A beam of neutral silver atoms produced by heating silver in an oven. It was collimated by a few fine slits and then passed through a non-homogeneous magnetic field in which the field increases in $z$-direction. The beam leaving the magnetic field was received on a photographic plate $P$.

**Observation :** Instead of a trace of direct beam, two were obtained symmetrically situated w.r.t. the direct beam as two discrete components, one in +*z*-direction and other in –*z*-direction.

(i) The intensity of both the trace is the same.

(ii) The results obtained were independent of the choice of the *z*-direction.

**Inference :**

A magnetic dipole experiences a net deflecting force in a non-homogeneous magnetic field which depends on the orientation of the magnet in the field. Since atoms are tiny magnetic dipoles, they experience deflecting forces when passing through the field. If an atom could have any orientation in the magnetic field then for millions of atoms present in the beam all possible orientations would be obtained and the beam would be deflected into a continuous band. In the experiment however, there was no band but discrete traces on the photographic plate. This showed that the atoms passing through the magnetic field were oriented in space in discrete directions so that beam deflected in certain discrete directions only and gave discrete traces on the plate.

Thus the experimental results are qualitatively direct proof of space quantisation of atoms.

## 10.3. Spectroscopic Terms: L-S and J-J Couplings

**1. Calculation of spectroscopic terms:** Spectroscopic term is one, where electron in an atom will more probably live.

The spectroscopic terms are determined as

$$\boxed{n^{2s+1}L_J^P}$$

where, n is principal quantum number, s is spin quantum number, L is orbital quantum number, J is total angular quantum number and P stands for parity.

If the spectroscopic term is obtained for a system of odd electronic configurtion, we write 'O' and for even electronic configuration we don't write anything (The odd and even configuration is determined by the total $l$ values).

If L = 0, S state ; L = 1, P state; L = 2, D state etc.

**2. Various types of coupling:**

There are two types of coupling, arises due to three kind of interactions viz. spin-spin correlation, orbital-orbital interaction (residual electrostatic interaction) and spin-orbit interaction. There are

(i) L-S coupling (in lighter atoms)

(ii) j-j coupling (in heavieur atoms)

**(i) L-S coupling:** This is also known as Russel-Saunder's coupling. This type of coupling happens in lighter atoms. In this coupling, spin-spin correlation and orbital-orbital interaction is stronger in comparision of spin-orbit interactions. It is divided into three steps:

**(a) Spin-spin correlation:** Due to this stronger interaction, the spin angular momentum of individual optical electrons combines to give a resultant spin angular momentum $(\vec{S})$, given as

$$\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3 + \ldots\ldots\ldots$$

The magnitude of resultant spin, angular momentum $\vec{S}$ is given by $|\vec{S}| = \hbar\sqrt{s(s+1)}$

Where the total spin quantum number (s) can take the values.

$$S = |s_1 + s_2 + \ldots|_{\min},\ |s_1 + s_2 + \ldots|_{\min} + 1, \ldots\ldots (s_1 + s_2 + \ldots\ldots)$$

**Example:** For one electron, $S = s_1 = \frac{1}{2}$ (doublet as 2s+1=2)

For two electrons, $s_1 = \frac{1}{2}, s_2 = \frac{1}{2}$

$\Rightarrow \quad S = |s_1 - s_2| ....... (s_1 + s_2)$

$= 0, 1$ (singlet, triplets)

For three electrons; $s_1=\frac{1}{2},\ s_2=\frac{1}{2},\ s_3=\frac{1}{2}$

$S'=0\ ,1$

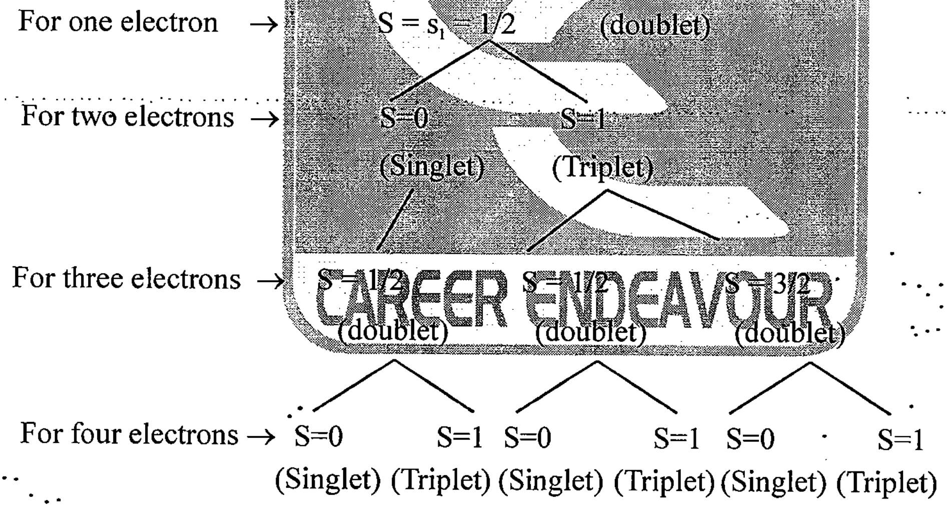
$S=\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ (Two doublets and one quartets)

In general,

**1.** For N even electrons → $S = 0, 1, 2 ........ N/2$

**2.** For N odd electrons → $S = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} ........ \frac{N}{2}$

**OR**


For one electron → S = s1 = 1/2 (doublet)
For two electrons → S=0 (Singlet) S=1 (Triplet)
For three electrons → S=1/2 (doublet) S=1/2 (doublet) S=3/2 (doublet)
For four electrons → S=0 S=1 S=0 S=1 S=0 S=1
(Singlet) (Triplet) (Singlet) (Triplet) (Singlet) (Triplet)

**(b) Residual electrostatic interaction:** As a result of this stronger interaction, the orbital angular moment of individual electrons coupled to give the resultant orbital angular momentum $\vec{L}$, given as:

$\vec{L} = \vec{\ell}_1 + \vec{\ell}_2 + \vec{\ell}_3 + ...........$

Magnitude given by, $|\vec{L}| = \hbar\sqrt{L(L+1)}$

Where L is known as total orbital angular momentum quantum number and it is given as

$L = |\vec{\ell}_1 + \vec{\ell}_2 + .....|_{\min}, |\vec{\ell}_1 + \vec{\ell}_2 + .....|_{\min} + 1, \ .....(\ell_1 + \ell_2 + .......)$

**Example:** For electron in 2p $\rightarrow \ell_1 = 1 = L$ ($P$ state)

For electrons in 3p 3d $\rightarrow \quad \ell_1 = 1, \ \ell_2 = 2$

$$\Rightarrow \quad L = (2-1).......(2+1)$$

$$= 1, 2, 3 \qquad (P, D, F \text{ state})$$

For electrons in 2p 3p 3d → $\underbrace{l_1 = 1 \qquad l_2 = 1}_{L' = 0,1,2} \qquad l_3 = 2$

$\left.\begin{matrix} L'=0 \\ l_3=2 \end{matrix}\right]$ L=2 (D state)

$\left.\begin{matrix} L'=1 \\ l_3=2 \end{matrix}\right]$ L=1,2,3 (P,D,F states)

$\left.\begin{matrix} L'=2 \\ l_3=2 \end{matrix}\right]$ L=(0,1,2,3,4) (S,P,D,F,G states)

**(c) Spin-orbit interaction:** As a result of this weak interaction, the resultant $\vec{S}$ & $\vec{L}$, couples less strongly to give total resultamt angular momentum $(\vec{J})$ given by $\vec{J} = \vec{L} + \vec{S}$ of magnitude $|\vec{J}| = \hbar\sqrt{J(J+1)}$, where total angular momentum quantum number J is given as

$$J = |L-S|_{\min}, \left(|L-S|_{\min} + 1\right), .........(L+S)$$

**(ii) j-j coupling:** This coupling is just opposite to that of L–S coupling. It happens in heavier atoms. In it spin-orbit interaction is relatively stronger than spin-spin correlation and redisual electrostatic interaction. It happens in the following steps:

**(a) Spin-orbit interaction:** As a result of this relatively stronger interaction the spin angular momentum of individual electrons combine with the orbital angular momentum to give the resultant angular momentum vectors $\vec{j}_1, \vec{j}_2$.........

$$\vec{j}_1 = \vec{\ell}_1 + \vec{s}_1, \vec{j}_2 = \vec{\ell}_2 + \vec{s}_2, ............ \qquad \text{where } |\vec{j}_1| = \hbar\sqrt{j_1(j_1+1)} \ \& \ j_1 = \ell_1 \pm s_1 = \ell_1 \pm \frac{1}{2}$$

**(b) Spin-spin correlation and residual electrostatic interaction:**

As a result of these weaker interactions, $\vec{j}_1, \vec{j}_2$..... combines to give the total resultant angular momentum $\vec{J}$ given by:

$$\vec{J} = \vec{j}_1 + \vec{j}_2 + .........$$

of magnitude $|\vec{J}| = \hbar\sqrt{J(J+1)}$

And $J = |\vec{j}_1 + \vec{j}_2 + ..........|_{\min}, |\vec{j}_1 + \vec{j}_2 + ......|_{\min} + 1, .....(j_1 + j_2 + ........)$

J is total angular momentum quantum number.

**Example:** For 4p 4d electrons: $\ell_1 = 1, S_1 = \frac{1}{2} \Rightarrow j_1 = \frac{1}{2}, \frac{3}{2}$ for p electron

$$\ell_2 = 2, S_2 = \frac{1}{2} \qquad \Rightarrow \qquad j_2 = \frac{3}{2}, \frac{5}{2} \text{ for D electron}$$

Now $(j_1, j_2)$ gives J

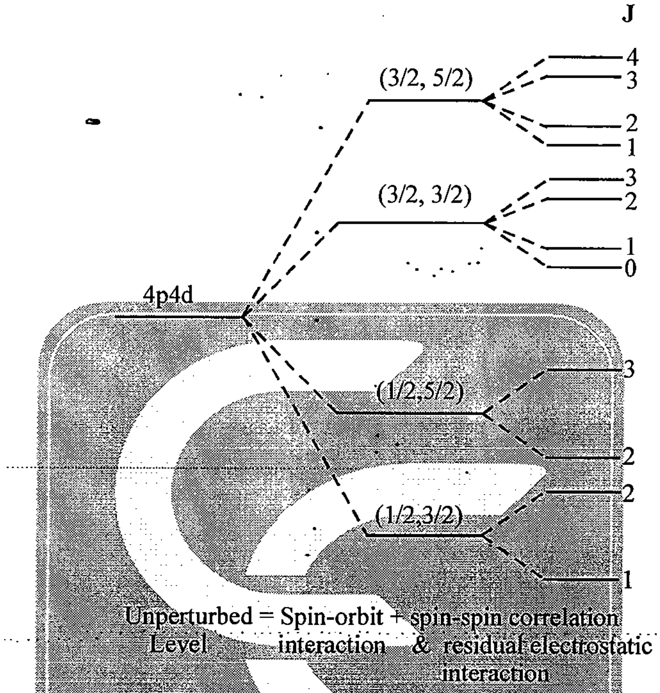
Values of $j_1$ & $j_2$ has 4 possible $(j_1, j_2)$ combinations

$\Rightarrow \quad \left(\frac{1}{2}, \frac{3}{2}\right)$ gives J=1,2

$\left(\frac{1}{2},\frac{5}{2}\right)$ gives J=2,3

$\left(\frac{3}{2},\frac{3}{2}\right)$ gives J=0,1,2,3

$\left(\frac{3}{2},\frac{5}{2}\right)$ gives J=1,2,3,4


J
4
3
2
1
3
2
1
0
3
2
2
1
(3/2, 5/2)
(3/2, 3/2)
4p4d
(1/2,5/2)
(1/2,3/2)
Unperturbed Level
= Spin-orbit interaction
+ spin-spin correlation & residual electrostatic interaction

**Determination of spectral terms under L-S coupling:**

Since the core electrons do not contribute to the angular momentum of the atom. We must consider only the (optical) electron outside the completed subshell for completed subshell for computing the term. We can divide the various atoms into three categories:

**(i) Atom with one optical electrons:**

For a hydrogen like atom ground state configuration is 1s for this we have

$$\Rightarrow S_1 = \frac{1}{2}, \ell_1 = 0$$

So, that $S = s_1 = \frac{1}{2}$

Multiplicity $\left(2\times\frac{1}{2}+1\right) = 2$ (doublet)

$L = \ell_1 = 0$ (S-state) and $j = |L-S|, \ldots\ldots. |L-S|+1 \ldots\ldots. (L+S) = \frac{1}{2}$

Thus the ground state term of H-atom is ${}^2S_{1/2}$. The ground state configuration of an alkali atoms Li, Na, K etc which have optical electron and electronic configuration $(n-1)s^2p^6ns^1$. The ground state term is ${}^2S_{1/2}$.

The excited state configuration and corresponding term for a H like atom would be

$2s, 3s, 4s$........... ${}^2S_{\frac{1}{2}}$

$2p, 3p, 4p$......... ${}^2p^0_{\frac{1}{2}}, {}^2p^0_{\frac{3}{2}}$ or ${}^2p^0_{\frac{1}{2},\frac{3}{2}}$

$3d, 4d$............... ${}^2D_{\frac{3}{2}}, {}^2D_{\frac{5}{2}}$

**(ii) Atoms with two or more non-equivalent optical electrons:**

Let us consider an atom having optical electrons with outer configuration (2s 2p), (3s 4s) (3s, 4p) (In this, $n$ & $\ell$ are not same, one of them is different or both are different).

Let us calculate terms for $\boxed{2p\,3d}$

For this we have $S_1 = \frac{1}{2}, S_2 = \frac{1}{2}, \ell_1 = 1, \ell_2 = 2$ the possible value of S & L are

$S = |S_1 - S_2|, |S_1 - S_2| + 1 ....... (S_1 + S_2)$

$S = 0, 1$

multiplicity $= (2s+1) = 1, 3$ (singlet, triplet)

and $L = |\ell_1 - \ell_2|, |\ell_1 - \ell_2| + 1 ........ |\ell_1 + \ell_2|$

$L = 1, 2, 3$ $(P, D, F$ states$)$

All the terms are odd because the configuration 4p 4d is odd $(\Sigma\ell = 1 + 2 = 3)$

Now for singlets terms we have $\begin{cases} S = 0, & J = 1 \\ L = 1 & \end{cases}$

$\Rightarrow$ term ${}^1P_1^0$

$\begin{cases} S = 0, & j = 2 \\ L = 2 & \end{cases} \Rightarrow$ term ${}^1D_2^0$

and $\begin{cases} S = 0, & j = 3 \\ L = 3 & \end{cases} \Rightarrow$ term ${}^1F_3^0$

For triplet terms, we have

$\begin{cases} S = 1, & j = 0, 1, 2 \\ L = 1 & \end{cases} \Rightarrow$ term ${}^3p_0^0, 3p_1^0, 3p_2^0$

$\begin{cases} S = 1, & j = 1, 2, 3 \\ L = 2 & \end{cases} \Rightarrow$ term $3D_1^0, 3D_2^0, 3D_3^0$

$\begin{cases} S = 1, & j = 2, 3, 4 \\ L = 3 & \end{cases} \Rightarrow$ term $3F_2^0\ 3F_3^0\ 3F_4^0$

For practice, certain outer electrons configuration and their term are given in table.

1. SS → ${}^1S_0$, ${}^3S_1$
2. SP → ${}^1P_1^0$, ${}^3P^0_{0,1,2}$
3. SD → ${}^1D_2, {}^3D_{1,2,3}$

4. PP $\rightarrow$ ${}^1S_0, {}^1P_1, {}^1D_2, {}^3S_1, {}^3P_{0,1,2}, {}^3D_{1,2,3}$

${}^3D_{1,2,3}, {}^3F_{2,3,4}, {}^3G_{3,4,5}$

**(i) For SS** $\rightarrow$ $S_1 = \frac{1}{2}, S_2 = \frac{1}{2}, \ell_1 = 0, \ell_2 = 0$

$S = 0,1$ multiplicity $(2s+1) = 1,3$ (singlet triplet)

For singlet $\rightarrow$ $s = 0, \quad J = 0$

$L = 0$

For triplet $S = 1, \quad J = 1$

$L = 0$ term ${}^3s_1$

**(ii) For SP:** $s_1 = \frac{1}{2}, s_2 = \frac{1}{2}, \ell_1 = 0, \ell_2 = 1$

$S = 0,1, j = 0,1,2$

term ${}^1p_0^1$

For triplet $S = 1 \quad J = 0,1,2$

$L = 1$ term ${}^3p_{0,1,2}^0$

**(iii) For SD:** $s_1 = \frac{1}{2}, s_2 = \frac{1}{2}, \ell_1 = 0, \ell_2 = 2$

$S = 0,1$

$L = 2$

For singlet $S = 0, \quad J = 2$

$L = 2$ term ${}^1D_2$

For triplet $S = 1 \quad J = 1,2,3$

$L = 2$ term ${}^3D_1, {}^3D_2, {}^3D_3$

OR

terms are ${}^1D_2, {}^3D_{1,2,3}$

**(iv) For PP:** $s_1 = \frac{1}{2}, s_2 = \frac{1}{2}, \ell_1 = 1, \ell_2 = 1$

$S = 0,1$

$L = 0,1,2$

For singlet $\begin{cases} S = 0 \\ L = 0 \end{cases}, \quad J = 0$

term ${}^1S_0$

$\begin{cases} S = 0 \\ L = 1 \end{cases}, \quad J = 1 \quad$ term ${}^1p_1$

$\begin{cases} S = 0 \\ L = 2 \end{cases}, \quad J = 2 \quad$ term ${}^1D_2$

For triplet $\begin{cases} S=1 \\ L=0 \end{cases}$, $J=1$

term $^3S_1$

$\begin{cases} S=1 \\ L=1 \end{cases}$, $J=0,1,2$ term $^3p_0, ^3p_1, ^3p_2$

$\begin{cases} S=1 \\ L=2 \end{cases}$, $J=1,2,3$

term $^3D_1, ^3D_2, ^3D_3$

Similarly, we can determine the spectroscopic terms for three or more non-equivalent electrons.

**(iii) Atoms with two or more equivalent electrons:**

For two equivalent electrons (same $n$ and $l$ values for both or more electrons) the values atleast one of the remaining quantum numbers $(m_\ell$ or $m_s)$ must differ to satisfy Pauli exclusion principle. Hence certain terms which were possible for two or more non equivalent electrons are now not possible here. Before the determination of terms, we must discuss two important factors:

**(i)** A closed subshell such as $s^2, p^6, d^{10}, f^{14}$ always form a $^1S_0$ term only since for them:

$\Sigma m_\ell = 0$

$\Sigma m_s = 0$

This means that $M_L = 0,\ M_s = 0$

and so, $L=0, S=0 \Rightarrow J=0 \rightarrow {}^1S_0$

(s-state) (singlet)

Therefore, J = 0, i.e. the only possible term is $^1S_0$. Hence, we conclude that when a subshell is completely filled the only allowed state is one in which the total spin angular momentum, Total orbital angular momentum and total angular momentum all are zero and hence no magnetic moment.

**(ii)** The term of configuration $(n\ell)^q$ are the same as the terms of the configuration $(n\ell)^{r-q}$ where $r$ is the max. numbers of electrons i.e. $2(2\ell+1)$.

**Example:**

Terms for $p^5$ the same as those of $p^{6-5}=p^1$ term

Terms for $p^4$ the same as those of $p^{6-4}=p^2$ the term

Terms for $d^8$ the same as those of $d^2$ and so on.

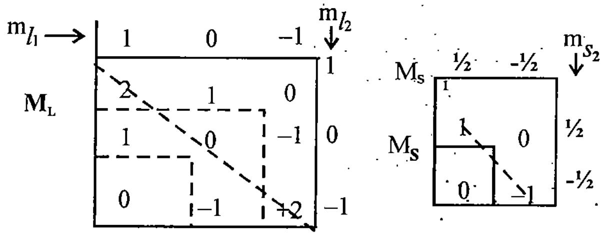
**Calculation of spectroscopic term for $p^2$:**

Here $\left.\begin{matrix} \ell_1 = 1;\ s_1 = \dfrac{1}{2} \\ \ell_2 = 1;\ s_2 = \dfrac{1}{2} \end{matrix}\right] \Rightarrow L = 0,1,2\ \&\ S = 0,1$

For the calculation of spectroscopic terms in case of equivalent electrons, we have to consider the Pauli's exclusion principle i.e the four quantum number $(n, \ell, m_l$ and $m_s)$ should not be same for any two electrons in an atom. For the calculation of spectroscopic terms in case equivalent electrons we use the Breit-Scheme which is follows as:

**Breit scheme for possible $M_L$-values is:**

**(i) For S =1**



$$\begin{array}{c|ccc|c} m_{l_1} \rightarrow & 1 & 0 & -1 & m_{l_2} \downarrow \\ \hline M_L & 2 & 1 & 0 & 1 \\ & 1 & 0 & -1 & 0 \\ & 0 & -1 & -2 & -1 \end{array} \qquad \begin{array}{c|cc|c} M_S & \frac{1}{2} & -\frac{1}{2} & m_{s_2} \downarrow \\ \hline M_S & 1 & 0 & \frac{1}{2} \\ & 0 & -1 & -\frac{1}{2} \end{array}$$

We take, only L = 1

We have to leave the diagonal values for the validity. Because of Pauli's exclusion principle

$\therefore \quad J = 0, 1, 2$

$\therefore$ terms are ${}^3p_{0,1,2}$

**(ii) For S = 0**

We take the remaining L values i.e.

L = 0, 2

For L = 0, S = 0 J = 0, term is ${}^1S_0$ Parity is even, as $\Sigma\ell$ for $p^2$ = even

For L = 2 S = 0 J = 2 term is ${}^1D_2$

**The terms for various electrons configurtion are given as:**

| Electonic configuration | Terms |
|---|---|
| $s^1$ | ${}^2s_{1/2}$ |
| $s^2, p^6, d^{10}, f^{14}$ | ${}^1s_0$ |
| $p^1, p^5$ | ${}^2p^0_{1/2,3/2}$ |
| $p^2, p^4$ | ${}^1s_0, {}^1D_2, {}^3p_{0,1,2}$ |
| $p^3$ | ${}^2p^0_{1/2,3/2}, {}^2D_{3/2,5/2}, {}^4S^0_{3/2}$ |
| $d^1, d^9$ | ${}^2D_{3/2,5/2}$ |
| $d^2, d^8$ | ${}^1S_0, {}^1D_2, {}^1G_4, {}^3P_{0,1,2}, {}^3F_{2,3,4}$ |

**Order of terms and fine structure levels:**

The relative energies of the various terms and levels which arise from a given (equivalent) electron configuration may be deduced from a set of rules given by Hund. These rules are:

(i) Of the terms arising from equivalent electron, those with largest multiplicity (2s+1) lie lowest.

(ii) Of the terms with given multiplicity and arising from equivalent electron, that with largest L, value lies lowest.

(iii) (a) In the multiplets formed from equivalent electron in a less than half-filled subshell, the level with lowest J, lies lowest (normal order).

(b) In the multiplets formed from equivalent electron in a more than half filled subshell, the level with highest J lies lowest (inverted order)

(c) Terms arising from half filled subshell show only very slight fine structure splitting.

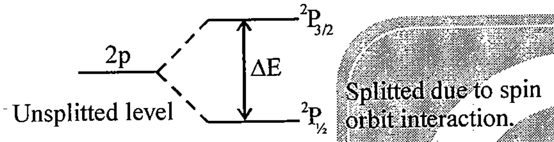
(d) The lowest terms arising from half filled subshell are the s-terms and are specially stable. These terms are ${}^2S_{1/2}$ for a half filled s-subshell, ${}^4S_{3/2}$ for a half-filled p-subshell, ${}^6S_{5/2}$ for a half filled d-subshell and ${}^8S_{7/2}$ for a half-filled f-subshell.

Hund was quite successful in predicting the lowest terms and levels of many atoms. However exceptions do occur:

**Example:**

| Atom | Electronic configuration | Terms |
|---|---|---|
| $_5B$ | $1s^2, 2s^2, 2p^1$ | $^2P^0_{1/2}$ (ground), $^2P^0_{3/2}$ |
| $_6C$ | $1s^2, 2s^2, 2p^2$ | $^1S_0, ^1D_2, ^3P_0$ (ground), $^3P_1, ^3P_2$ |
| $_7N$ | $1s^2, 2s^2, 2p^3$ | $^2p^0_{1/2}, ^2p^0_{3/2}, ^2D_{3/2}, ^2D^0_{5/2}, ^4s^0_{3/2}$ (ground) |
| $_8O$ | $1s^2, 2s^2 2p^4$ | $^1S_0, ^1D_2, ^3P_0, ^3P_1, ^3P_2$ (ground) |
| $_9F$ | $1s^2, 2s^2, 2p^5$ | $^2P^0_{1/2}, ^2P^0_{3/2}$ (ground) |
| $_{17}Cl$ | $1s^2, 2s^2, 2p^6, 3s^2, 3p^5$ | $^2P^0_{1/2}, ^2P^0_{3/2}$ (ground) |

**Spin-Orbit Interaction**



The term shift in cm$^{-1}$,

$$\Delta T = -\frac{\Delta E}{hc} = 5.84\frac{Z^4}{n^3\ell(\ell+1)}cm^{-1}$$

Or, $\Delta T \propto Z^4$.

## 10.4. Spectra of Alkali Elements

**Broad feature of Alkali Spectra:**

After H-atom, the simplest atoms to study with regard to their spectra are the 'Alkali atoms' Li, Na, K, Rb, Cs & Fr. The experimental study of their spectra was taken up by living, Dewar, Rydberg and Bergmann spectral lines in the emission spectrum of an alkali atom formed four series a principal series of bright and most presistant lines, a sharp series of fine lines a diffuse series a comparatively broader lines and fundamental (or Bergmann) series in the infrared region.

Taking clue from of the Balmer's formula for H.

$$\boxed{v^B_m = v^B_\infty - \frac{R}{n^2}}$$

Rydberg pointed out that the alkali series could be represented by similar formula such as

(i) Principal : $v^P_m = v^P_\infty - \frac{R}{(m+p)^2}$; $m = 2, 3, 4.......\infty$

(ii) Sharp : $v^s_m = v^s_\infty - \frac{R}{(m+s)^2}$; $m = 2, 3, 4..........\infty$

(iii) Diffuse : $v^D_m = v^D_\infty - \frac{R}{(m+d)^2}$; $m = 3, 4, 5.........\infty$

(iv) Fundamental : $v_m^f = v_m^f - \dfrac{R}{(m+f)^2}$; $m = 3, 4, 5......\infty$

where $v_\infty^s$ are the wave number of convergence limits of the corresponding series and are called 'fixed terms'. In the other so called running terms. The $p, s, d$ and $f$ represent Rydberg corrections for the corresponding series.

Rydberg noticed the following relations among different series of the same atom. These are

(i) The sharp and diffuse series have a common convergence limit $\left[v_\infty^s = v_\infty^d\right]$

(ii) The common convergence limit is equal to the first running term (with m = 2) of the principal series

$$\left[v_\infty^s = v_\infty^d = \frac{R}{(2+p)^2}\right]$$

(iii) The convergence limit of the principal series is equal to the running term (with m = 1) of the sharp series.

$$\left[v_\infty^p = \frac{R}{(1+s)^2}\right]$$

(iv) The convergence limit of the fundamental series is equal to the first running term (with m = 3) of the diffuse series).

$$\left[v_\infty^f = \frac{R}{(3+d)^2}\right]$$

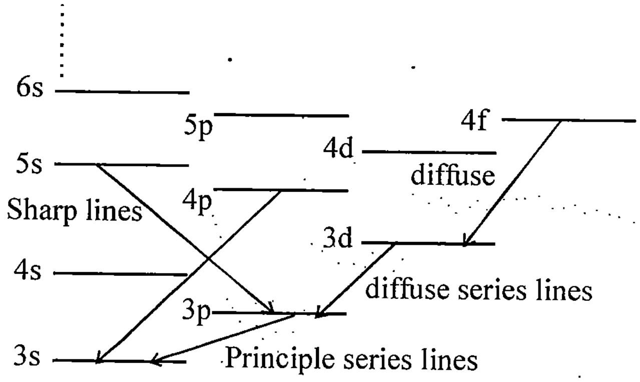
In view of these observations the above formula may be written in the following form:

(i) Principal : $v_m^p = \dfrac{R}{(1+s)^2} - \dfrac{R}{(m+p)^2}$; $m = 2, 3, 4....\infty$

(ii) Sharp : $v_m^s = \dfrac{R}{(2+p)^2} - \dfrac{R}{(m+s)^2}$; $m = 2, 3, 4....\infty$

(iii) Diffuse : $v_m^d = \dfrac{R}{(2+p)^2} - \dfrac{R}{(m+d)^2}$; $m = 4, 5, ....\infty$

(iv) Fundamental : $v_m^f = \dfrac{R}{(3+d)^2} - \dfrac{1}{(m+1)^2}$; $m = 4, 5, ....\infty$



[Emission spectra of sodium]

## 10.5. Zeeman Effect

The splitting of spectral lines by a magnetic field is called Zeeman effect. It was discovered by Zeeman in 1896. It is a configuration of space quantisation. When an atom is placed in a magnetic field, its spectral lines further split into several polarised components. The splitting is proportional to the strength of the applied field.

### 1. Normal Zeeman effect :

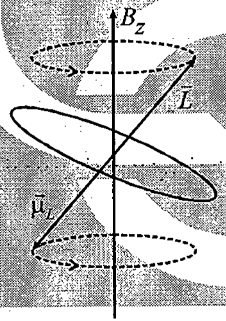
The normal Zeeman effect which is shown by all lines due to transitions between the singlet ($s=0$) states of an atom, can be explained from the classical theory and also from the quantum theory without taking note of electronic spin.

In the normal Zeeman effect a single line of wavelength $\lambda_0$ in the presence of a magnetic field is split into three components with wavelengths $\lambda_0-\Delta\lambda$, $\lambda_0$ and $\lambda_0+\Delta\lambda$ when viewed at right angles to the magnetic field $\vec{B}$.

The lines of wavelength $\lambda_0+\Delta\lambda_0$ are plane-polarised with electric vector perpendicular to $\vec{B}$, while the central line has the wavelength of the source and is plane polarized with electric vector parallel to the field $\vec{B}$. This is called a "normal triplet".

**Explanation :** In terms of quantum theory, an atom possesses an angular polyelectronic momentum $\vec{L}$ and an orbital magnetic moment $\vec{\mu}_L$ with following relation

$$\frac{|\vec{\mu}_L|}{|\vec{L}|}=\frac{e}{2m} \quad (\vec{\mu}_L \text{ is directed opposite to } \vec{L})$$



The magnetic potential energy of an atom in a magnetic field is given by

$$E=-\vec{\mu}_L\cdot\vec{B}=-\mu_{L_z}B_z \quad (\because\ B \text{ is in } z\text{-direction})$$

$$\therefore\ E=-\left(-\frac{e}{2m}m_L\hbar\right)B_z=\left(\frac{eB\hbar}{2m}\right)m_L=m_L\left(\frac{e\hbar}{2m}\right)B \quad (\because\ B_z=B)$$

$\boxed{E=m_L\,\mu_B\cdot B}$, where $\mu_B=\dfrac{e\hbar}{2m}$ (Bohr magneton)

Because $m_L$ can have $2L+1$ values, a state of given orbital quantum number $L$ is split into $2L+1$ substates that differ in energy by $\mu_B\cdot B$ when the atom is in magnetic field. But the selection rules permit only.

$\boxed{\Delta m_L=0,\pm 1}$

So only three components

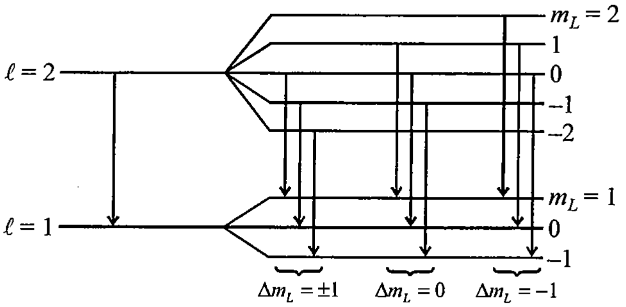
$\boxed{\nu_i=\nu_0-\dfrac{\Delta E}{h}}$ $(\Delta E=\Delta m_L\,\mu_B\cdot B)$

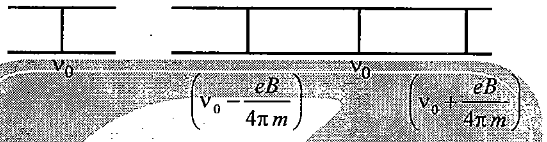
**Note:** $\boxed{\Delta=\text{final}-\text{initial}}$

$$\nu_1 = \nu_0 - \frac{\mu_B \cdot B}{\hbar} \qquad (\text{for } \Delta m_L = \pm 1)$$

$$\nu_2 = \nu_0 \qquad (\text{for } \Delta m_L = 0)$$

$$\nu_3 = \nu_0 + \frac{\mu_B \cdot B}{h} \qquad (\text{for } \Delta m_L = -1)$$


$\ell = 2$
$\ell = 1$
$m_L = 2$
1
0
−1
−2
$m_L = 1$
0
−1
$\Delta m_L = \pm 1$
$\Delta m_L = 0$
$\Delta m_L = -1$


$\nu_0$
$\nu_0$
$\left(\nu_0 - \frac{eB}{4\pi m}\right)$
$\left(\nu_0 + \frac{eB}{4\pi m}\right)$

(Spectrum without magnetic field) (Spectrum with magnetic field present)

Quantum mechanics leads not only to selection rules but also to the polarisation rules. For observation normal to the field, the transition $\Delta m_L = 0$ results in spectral line polarised with the electric vector parallel to the field ($\pi$-component) while the transition $\Delta m_L = \pm 1$ results in lines polarized with the electric vector perpendicular to the field ($\sigma$-component).

The separation of the Zeeman components is

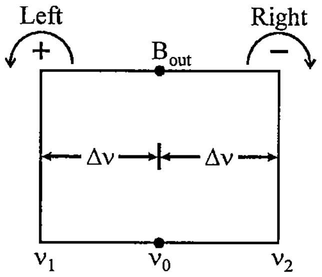
$$\boxed{\Delta \nu = \frac{eB}{4\pi m}} \quad \text{or} \quad \boxed{\Delta \lambda = \frac{\lambda^2 \Delta \nu}{c} = \frac{eB\lambda^2}{4\pi mc}}$$

We can find $e/m$ from $\boxed{\frac{e}{m} = \frac{4\pi \Delta \nu}{B}}$

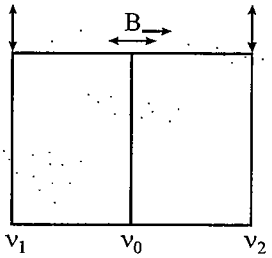
Normal transverse Zeeman effect and longitudinal Zeeman effect.

**(a) Transverse Zeeman effect:** when the source is viewed at right angles to magnetic field direction, it splits into three components one $\pi$ and two $\sigma$. This is called transverse Zeeman effect.

**(b) Longitudinal Zeeman effect:** when source is viewed in the direction of the magnetic field, along the z-axis, we obtain two lines which are circularly polarized in opposite sense. It is called longitudinal Zeeman effect.


Left
+
Right
−
$B_{out}$
$\Delta\nu$
$\Delta\nu$
$\nu_1$
$\nu_0$
$\nu_2$

(Viewed in direction of the field $\vec{B}$)
(Longitudinal Zeeman effect)


B
$\nu_1$
$\nu_0$
$\nu_2$

(Viewed perpendicular to field)
(Transverse Zeeman effect)

**2. Anomalous Zeeman Effect :**

When an atom is placed in an external magnetic field and then excited, the spectral line it emits in the de-excitation process are split into several components. This is called *Anomalous Zeeman Effect.*

Under the effect of external magnetic field, the magnetic moment associated with the total angular momentum $\vec{J}$ where $(\vec{J} = \vec{L} + \vec{S})$ ; interacts with the external magnetic field and the degeneracy in the energy due to the quantum number $J$ is broken.

The energy level corresponding to a particular $J$, splits up into $(2J+1)$ levels, and the magnitude of the splitting is given by $\Delta E = -\vec{\mu}_J \cdot \vec{B}$.

Before the advent of quantum theory, these splittings were divided into two categories: Normal and anomalous.

**3. Normal Zeeman effect:**

It is a special case of anomalous Zeeman effect where the transition takes place between the singlet states only. In this case, since $s = 0$ therefore, $J = L$ and $\vec{\mu}_J$ and $\vec{J}$ are anti-parallel. This results in equispaced energy splitting and we get less number of lines compared to anomalous Zeeman effect.

The precession of component of $\vec{\mu}$ perpendicular to $\vec{J}$ averages to zero. Only the component antiparallel to $\vec{J}$ remains a constant of magnitude $|\vec{\mu}_J|$ which contributes to the magnetic moment of the atom. Thus,

$$\vec{\mu}_J = \frac{(\vec{\mu} \cdot \vec{J})}{|\vec{J}|} \frac{\vec{J}}{|\vec{J}|}$$

$$\vec{\mu}_J = -\frac{\mu_B}{\hbar} \frac{\{(\vec{L}+2\vec{S}) \cdot (\vec{L}+\vec{S})\} \vec{J}}{|\vec{J}|^2} \qquad \text{[from equation (iii)]}$$

$$\text{and } \Delta E = -\vec{\mu}_J \cdot \vec{B} = +\frac{\mu_B}{\hbar} \left\{ \frac{(\vec{L}+2\vec{S}) \cdot (\vec{L}+\vec{S})}{|\vec{J}|^2} \right\} (\vec{J} \cdot \vec{B})$$

$$\Delta E = \frac{\mu_B}{\hbar} \left\{ \frac{L^2 + 2S^2 + 3\vec{L} \cdot \vec{S}}{|\vec{J}|^2} \right\} J_z B \qquad \text{... (iv)} \qquad \text{(since direction of } B \text{ is in } z\text{-direction)}$$

Since, $\vec{J} = \vec{L} + \vec{S}$ or $J^2 = L^2 + S^2 + 2\vec{L} \cdot \vec{S}$

$$\text{or } \vec{L} \cdot \vec{S} = \frac{J^2 - L^2 - S^2}{2} \qquad \text{... (v)}$$

Putting this in equation (iv), we have

$$\Delta E = \frac{\mu_B}{\hbar} \left\{ L^2 + 2S^2 + \frac{3(J^2 - L^2 - S^2)}{2} \right\} \frac{J_z B}{|\vec{J}|^2}$$

$$\text{So, } \Delta E = \frac{\mu_B}{\hbar} \left\{ \frac{3J^2 + S^2 - L^2}{2J^2} \right\} J_z B = \frac{\mu_B}{\hbar} \left\{ 1 + \frac{J^2 + S^2 - L^2}{2J^2} \right\} J_z B$$

Since, $J = \sqrt{j(j+1)}\,\hbar$, $L = \sqrt{\ell(\ell+1)}\,\hbar$, $S = \sqrt{s(s+1)}\,\hbar$ and $J_z = m_j \hbar$

Therefore, $\Delta E = \mu_B \left\{1 + \frac{j(j+1) + s(s+1) - \ell(\ell+1)}{2j(j+1)}\right\} B m_j$

Therefore, $\Delta E = \mu_B B g m_j$ ... (vi)

where $g = 1 + \frac{j(j+1) + s(s+1) - \ell(\ell+1)}{2j(j+1)}$ ... (vii)

This quantity $g$ is called "Lande's $g$-factor". We can see from the above equation (vii), that when $s = 0$, $j = \ell$ and $g = 1$ and $\Delta E$ from equation (vi) becomes $\Delta E = \mu_B B m_j$ and hence we get equal splitting proportional to $m_j$. While for the case $s \neq 0$, the $g$-factor will vary and we get unequal splittings.

**4. Applications of Zeeman Effect:**

(i) By analyzing the Zeeman splitting of the spectral lines of an atom, the spectroscopist determines the Zeeman splittings of the energy levels of the atom. These can conclusively confirm the assignment of quantum number $j$ to each level.

(ii) The $g$ values of the energy levels can be deduced from the splitting for a known applied field, giving information about the $L$ and $S$ values of the electron undergoing transitions. Overall, then the term symbols for various atomic states can be deduced by Zeeman experiments.

Much of what is known about the stars has come from measurements and analysis of optical lines spectra. The pattern of lines observed in emission spectra is used to identify the composition of stars; the intensity of lines observed in absorption spectra is used to measure the temperatures of stellar surfaces; the Doppler shift of the spectral lines is used to measure the velocity of stars and the Zeeman effect is used to measure the magnetic fields produced by stars. Generally polarization of spectra ($\sigma$ and $\pi$) is used to separate out components.

**Example:** Zeeman splitting of $D_1$ and $D_2$ lines of sodium.

$^2P_{3/2} \rightarrow {}^2S_{1/2}$ $D_2$ line

$^2P_{1/2} \rightarrow {}^2S_{1/2}$ $D_1$ line

The selection rule for transitions is $\Delta m_j = 0, \pm 1$ (but not $m_j = 0$ to $m_j = 0$ if $\Delta j = 0$)

| Terms | No. of Zeeman levels | $g$ | $m_j(+J,...,-J)$ | $\Delta E = g\, m_j\, \mu_B B\,(g\, m_j)$ |
|---|---|---|---|---|
| $^2S_{1/2}$ | 2 | 2 | $\pm\frac{1}{2}$ | $\pm 1$ |
| $^2P_{1/2}$ | 2 | $\frac{2}{3}$ | $\pm\frac{1}{2}$ | $\pm\frac{1}{3}$ |
| $^2P_{3/2}$ | 4 | $\frac{4}{3}$ | $\pm\frac{3}{2}, \pm\frac{1}{2}$ | $\pm 2, \pm\frac{2}{3}$ |

For $\Delta m_j = 0$; components are polarised with electric vector parallel to the magnetic field ($\pi$-components)

For $\Delta m_j = \pm 1$; components are polarised with electric vector parallel to the field ($\sigma$-components).

**5. Calculation of shift in energy:**

Since $\Delta E = \mu_B B m_j g$

(a) For $D_1$ transition (i.e., $^2P_{1/2} \rightarrow {}^2S_{1/2}$)

(i) Transition line $m_j = -\frac{1}{2}$ to $m_j = \frac{1}{2}$ line (a)

$$E_2 = E_i - E_f = E_0 + \frac{2}{3} \times \left(-\frac{1}{2}\right) \mu_B B - 2 \times \frac{1}{2} \mu_B B$$

$$E_1 = E_0 - \frac{4}{3} \mu_B B$$

(ii) Transition line $m_j = -\frac{1}{2}$ to $m_j = -\frac{1}{2}$ line (c)

$$E_2 = E_0 + \frac{2}{3} \times \left(-\frac{1}{2}\right) \mu_B B - 2 \times \left(-\frac{1}{2}\right) \mu_B B = E_0 + \frac{2}{3} \mu_B B$$

(iii) Transition line $m_j = \frac{1}{2}$ to $m_j = \frac{1}{2}$ line (b)

$$E_3 = E_0 + \frac{2}{3} \times \frac{1}{2} \mu_B B - 2 \times \left(-\frac{1}{2}\right) \mu_B B$$

$$E_3 = E_0 - \frac{2}{3} \mu_B B$$

(iv) Transition line $m_j = \frac{1}{2}$ to $m_j = -\frac{1}{2}$ line (d)

$$E_4 = E_0 + \frac{2}{3} \times \frac{1}{2} \mu_B B - 2 \times \left(-\frac{1}{2}\right) \mu_B B$$

$$E_4 = E_0 + \frac{4}{3} \mu_B B$$

So, energies are in the order $E_4 > E_2 > E_3 > E_1$

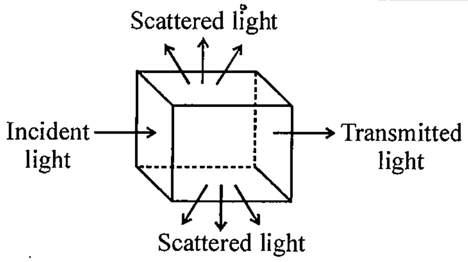
Zeeman observed the effect of external magnetic field ($\left(\sim \frac{1}{10} \text{ Tesla}\right)$ on spectrum lines and he observed that the spectrum line is splitted into a number of components. The electric vector of some of the lines is parallel to the applied magnetic field ($\pi$-component) while the electric field vector of some of the lines is transverse to the external magnetic field and are called $\sigma$-components.

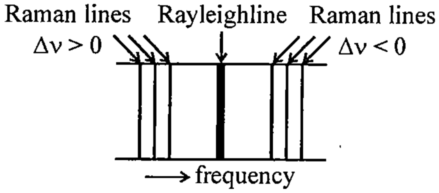
## 10.6. Raman Spectra

Discovered by C.V. Raman in 1928 and won nobel prize in 1930.

**1. Raman Effect :**

If a beam of monochromatic light is passed through the substance, nearly all of it is transmitted and a very small fraction of it is scattered in all directions. If the scattered light in a direction perpendicular to the incident beam is spectrographed, it is found not only to contain a line corresponding to the frequency of incident light, but also a pattern of relatively, weak lines on the low frequency side of the incident light and a similar pattern of still more weak lines on the high frequency side.





The line at the incident frequency is known as "Rayleigh line". The other weak lines are called the Raman lines. The difference between the frequencies of the Rayleigh line and a Raman line is known as the "Raman shift", and may be expressed as $\Delta\bar{\nu} = |\bar{\nu}_0 - \bar{\nu}_R|$, where the subscripts 0 and $R$ stands for incident and Raman lines, respectively.

For a given substance, the Raman shift is found to be independent of the frequency of the incident radiation, i.e., it has the same value for the different frequency of the incident light. The difference is a characteristic of the substance and is found to be equal to its vibrational or rotational transition.

**2. Nature of Raman Spectra: Molecular spectra**

Wave number of Rayleigh (excitting line or incident monochromatic light) = $\boxed{\nu_0 = \frac{1}{\lambda_0}}\ cm^{-1}$

Wave number of stokes Raman line = $(\nu_0 - \Delta\nu)\,cm^{-1}$ and wave number of anti-stokes Raman line $= (\nu_0 + \Delta\nu)\,cm^{-1}$, $\Delta\nu$ is fixed for a susbtance.

Intensity of Rayleigh line > Intensity of Stokes line > Intensity of anti-Stokes Raman line

The change in the energy of the molecule is either of vibrational energy or rotational energy. Therefore, Raman spectra is of two kind.

(i) Vibrational Raman spectrum

(ii) Pure rotational Raman spectrum.

**3. Quantum theory of Raman effect :**

The occurence of Raman scattering may be most easily by considering a radiation of frequency $\nu$ as consisting of a stream of particles called photons having energy $h\nu$. Photons can be imagined to undergo collisions with molecules and, if the collision is perfectly elastic, they will be deflected without any change in energy. A detector placed to collect energy at right angles to an incident beam will thus receive photons of frequency $\nu$. This gives rise to "Rayleigh scattering".

On the other hand, if the collision is inelastic, there can occur an exchange of energy between molecule and photon. This exchange of energy can be of two types.

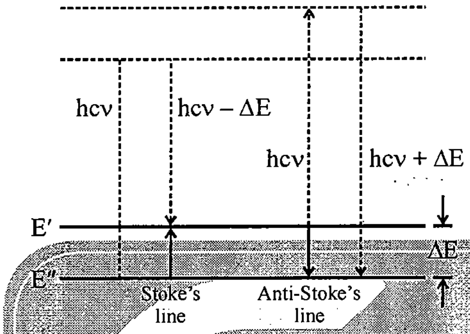
- The molecule before the collision occupies a lower vibrational or rotational energy level and after the emission of the photon, it may be present in any one of the higher energy levels as permitted by selection rule. In this case, the molecule gains energy equivalent to the energy difference between the two involved levels. This energy is provided by the photon and hence its energy is decreased and thus a Raman line on the low frequency side is observed. Such a Raman line is called a "Stokes" line.
- The molecule before the collision occupies one of the higher vibrational or rotational energy level and after the emission of the photon may be present in any one of the lower energy levels as permitted by the selection rule. In this case, the molecule loses an energy equivalent to the energy difference between the two involved levels. This, energy is gained by photon and hence its energy is increased and thus a Raman line on high frequency side is observed. Such a Raman line is called an "anti-Stokes" line.

**Stokes line is generally more intense than anti-stokes line.**

The Stokes lines frequency is $\dfrac{hc\nu - \Delta E}{hc}$.

The anti-stokes lines frequency is $\dfrac{hc\nu + \Delta E}{hc}$.

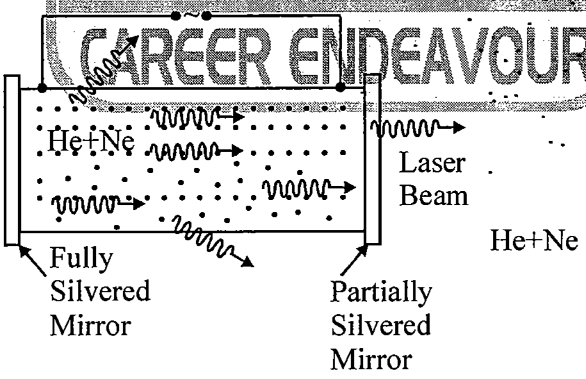
The wave number shift in each case being $\dfrac{\Delta E}{hc}$.



## 10.7. The Helium-Neon Laser (Four level Laser):

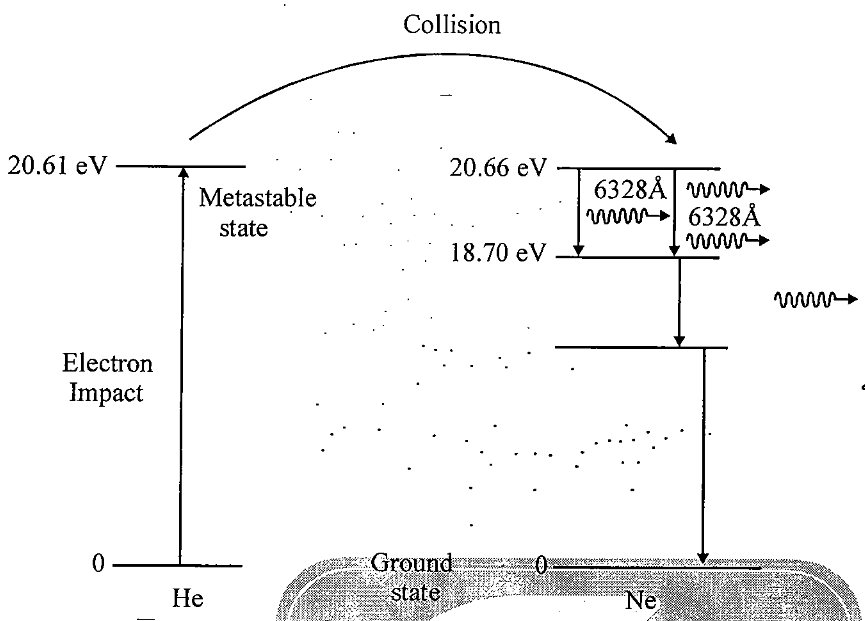
The Helium-Neon laser was the first gas laser to be operated successfully. It was fabricated by Javan in 1961. It is a four-level laser in which population inversion is achieved by electric discharge.

The Helium-Neion laser consists of a long and narrow discharge tube filled with a mixture of He and Ne in a ratio of about 10:1 at a pressure of about 1 mm of mercury. The gas mixtrue (He+Ne) forms the laser medium.



It is placed between a pair of optically plane and parallel mirrors which form a resonant cavity. One of the mirrors is fully silvered and the other is partially silvered. The spacing of mirrors is equal to an integral number of half-wavelengths of the laser light. An electric discharge may be produced in the gas-mixture by electrodes connected to a high-frequency electric source.

**Working:** When a discharge through the gas mixture, electrons are accelerated down the tube. These accelerated electrons collide with, and "pump" (excite) the He and Ne atoms to metastable states 20.61 eV and 20.66 eV respectively above their ground states (Figure below). Some of the excited He atoms transfer their energy to ground-state


Collision
20.61 eV
Metastable state
Electron Impact
0
He
20.66 eV
6328Å
6328Å
18.70 eV
0
Ground state
Ne

**A four-level laser**

Ne atoms by collision, with the 0.05 eV of addition energy being provided by the kinetic energy of atoms. Thus, He atoms help in achieving a population inversion in the Ne atoms.

When an excited Ne atom passes spontaneously from the metastable state at 20.66 eV to be the state at 18.70 eV, it emits a 6328 Å photon. This photon travels through the gas-mixture, and if it is moving parallel to the axis of the tube, is reflected back and forth by the mirror-ends until it stimulates an excited Ne atom and causes it to emit a fresh 6328 Å photon in phase with the stimulating photon. This process is continued and a beam of coherent radiation builds up in the tube. When this red-light becomes sufficiently intense, a portion of it escapes through the partially-silvered end.

## SOLVED PROBLEMS

1. When the Zeeman lines are observed perpendicular to the magnetic field, they will be
(a) plane polarized (b) circularly polarized
(c) elliptically polarized (d) unpolarized **[B.H.U-2010]**

**Soln.** When the Zeeman lines are observed perpendicular to the magnetic field, they will be plane polarised.

For $\Delta m_J = \pm 1 (\sigma \text{ lines})$ lines are plane polarised perpendicular to $\vec{B}$ direction.

And for $\Delta m_J = 0 (\pi \text{ lines})$ lines are plane polarised parallel to $\vec{B}$.

**Correct option is (a)**

2. Stern-Gerlach apparatus breaks up the atomic beam of silver atoms into following number of components
(a) One (b) Two (c) Three (d) Four **[B.H.U-2010]**

**Soln.** In Stern-Gerlach experiment, a beam of silver atoms is passed through an inhomogeneous magnetic field. In an inhomogeneous magnetic field, the force on the magnetic moment μ will be $\mu_z x$ (Gradient of the magnetic field in the z-direction) where $\mu_z = \pm \frac{eh}{4m_e}$, where $e$ is the charge of the electron, $m_e$ is its mass, $h$ is Planck's constant, and $z$ is the field direction.

Thus, depending on the orientaton of the magnetic moment relative to the magnetic field there will be either an attractive or repulsive force and the beam will split in two components, exhibiting spatial quantization.

**Correct option is (b)**

3. Which of the following sets of quantum numbers is not possible? **[B.H.U-2014]**
(a) $n = 2, l = 3, m_1 = 2$ (b) $n = 3, l = 2, m_1 = -2$
(c) $n = 6, l = 2, m_1 = 0$ (d) $n = 7, l = 3, m_1 = -3$

**Soln.** **Option (a):** $n = 2, \ell = 3, m_\ell = 2$

The values of $l$ lies between 0 and $n-1$, including 0 and $n-1$.

$l$ cannot be greator than $n$.

∴ option (a) is incorrect

**Option (b):** $n = 3, \ell = 2, m_\ell = -2$

For $n = 3$

Possible values of $\ell = 0, 1, 2$

Possible values of $m_\ell$ (corresponding to $\ell = 2$) $= -2, -1, 0, 1, 2$

∴ option (b) is correct

**Option (c):** $n = 6, \ell = 2, m_\ell = 0$

Possible values of $\ell = 0, 1, 2, 3, 4, 5$

Possible value of $m_\ell$ (corresponding to $\ell = 0$) $= 0$

∴ option (c) is correct.

**Option (d):** $n = 7, \ell = 3, m_\ell = -3$

Possible values of $\ell = 0, 1, 2, 3, 4, 5, 6$

Possible values of $m_\ell$ (corresponding to $\ell = 3$) $= -3, -2, -1, 0, 1, 2, 3$

∴ option (d) is correct.

**Correct option is (a)**

4. An atomic orbital with principle quantum number n can accommodate N number of electrons, which of the following statements is false ? **[B.H.U-2014]**

(a) $n^2 = N$ (b) $2n^2 = N$ (c) $N = \sum_{l=0}^{n-1} 2(2l+1)$ (d) $N = 2[1+3+5+\cdots 2n-1]$

**Soln.** An atomic orbital with principle quantum number $n$ can accomodate $2n^2$ number of electrons.

$\therefore$ option (b) is correct.

option (c) and (d) are same. Total number of electrons in an atomic orbital is given by

$$N = \sum_{\ell=0}^{n-1} 2(2\ell+1) = 2\left[1+3+5+\cdots+(2n-1)\right]$$

$\therefore$ Options (c) and (d) are correct.

Option (a) is false as $N \neq n^2$.

**Correct option is (a)**

5. If the outer orbit of an atom contains two electrons, the possible multiplicity would be

(a) 1, 3 (b) 1 (c) 3 (d) 0 **[B.H.U-2016]**

**Soln.** For an atom with its outer orbit having two electrons can have two configuration :

(i) Singlet state : opposite spins, therefore total spin S = 0
multiplicity = 2S + 1 = 2 × 0 + 1 = 1

(ii) Triplet state : same spin, total spin = 1
multiplicity = 2S + 1 = 2 × 1 + 1 = 3

**Correct option is (a)**

6. For an electron with the quantum number $l = 2$, the possible values of $j$ are **[B.H.U-2016]**

(a) $\frac{3}{2}, \frac{3}{2}$ (b) $\frac{5}{2}, \frac{3}{2}$ (c) $\frac{5}{2}, \frac{3}{2}, \frac{1}{2}$ (d) $\frac{3}{2}, \frac{1}{2}$

**Soln.** Given: for an electron with quantum number $\ell = 2$,

$$j = |\ell - s| \text{ to } |\ell + s|$$

$$= \left|2 - \frac{1}{2}\right| \text{ to } \left|2 + \frac{1}{2}\right| = \frac{3}{2}, \frac{5}{2}$$

**Correct option is (b)**

7. The ground state of nitrogen atom would be

(a) ${}^4S_{3/2}$ (b) ${}^2D_{3/2}$ (c) ${}^2D_{5/2}$ (d) ${}^2P_{3/2}$ **[B.H.U-2010]**

**Soln.** Atomic number of nitrogen atom : $Z = 7$

Electronic configuration of Nitrogen : $1s^2 2s^2 2p^3$

$2p^3$ is half-filled, so these electrons will contribute to the total angular momentum.

$$\begin{array}{cccc} m_l & 1 & 0 & -1 \\ & \boxed{\uparrow} & \boxed{\uparrow} & \boxed{\uparrow} \\ m_s & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}$$

$$M_L = \sum m_\ell = 0 \Rightarrow L = 0$$

$$M_S = \sum m_s = \frac{3}{2} \Rightarrow S = \frac{3}{2}$$

$$J = L + S = \frac{3}{2}$$

Term symbol: ${}^{2S+1}L_J \Rightarrow {}^4S_{3/2}$

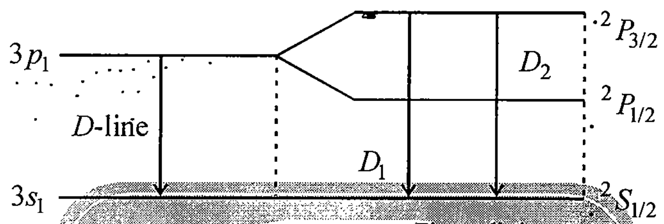
**Correct option is (a)**

8. The two sodium D-lines have wavelengths 5890 and 5896 Å. These arise due to transition from ${}^2S_{1/2}$ to
(a) ${}^2P_{3/2}$ and ${}^2P_{1/2}$ (b) ${}^2P_{1/2}$ and ${}^2P_{3/2}$ (c) ${}^2D_{5/2}$ and ${}^2D_{3/2}$ (d) ${}^2D_{3/2}$ and ${}^2D_{5/2}$ **[B.H.U-2013]**

**Soln.** Transitions in Zeeman splitting for Na D-line

$Na = 1s^2 2s^2 2p^6 3s^1$ – Ground state

$Na^* = 1s^2 2s^2 2p^6 3p^1$ – Excited state



Fine splitting

$\Delta L = \pm 1$

$\Delta S = 0, S = \frac{1}{2}$

$\Delta J = 0, \pm 1$

For $P$ electron, $S = \frac{1}{2}, J = \frac{1}{2}, \frac{3}{2} \Rightarrow {}^2P_{1/2}, {}^2P_{3/2}$

For $S$ electron, $L = 0, S = \frac{1}{2}, J = \frac{1}{2} \Rightarrow {}^2S_{1/2}$

$D_1 = 5896\ \text{Å} : {}^2P_{1/2} \rightarrow {}^2S_{1/2}$

$D_2 = 5896\ \text{Å} : {}^2P_{3/2} \rightarrow {}^2S_{1/2}$

**Correct option is (a)**

9. In a He-Ne laser, the key factor in the process of population inversion is **[B.H.U-2016]**
(a) The energy transfer from He to Ne (b) The energy transfer from Ne to He
(c) The material state of Ne (d) The collision of electron with the wall

**Soln.** In He-Ne laser, the function of He is to provide the assistance in the population inversion. The energy transfer from He to Ne occurs by collision and is thus the key factor in the process of population inversion.

**Correct option is (a)**

10. Principal series arises due to transitions **[B.H.U-2015]**
(a) $nf \rightarrow 3d, n > 3$ (b) $nd \rightarrow 3p, n > 2$ (c) $ns \rightarrow 3p, n > 3$ (d) $np \rightarrow 3s, n > 2$

**Soln.** The principal series arises due to transitions

$np \rightarrow 3s$, where $n > 2$ .

The sharp series arises due to transitions

$ns \rightarrow 3p, n > 3$

The diffuse series arises due to transitions

$nd \to 3p, n > 2$

The foundamental series arises due to transitions

$nf \to 3d, n > 3$

**Correct option is (d)**

11. If the existing line in the Raman effect experiment is 5460 Å and the Stokes' line is at 5520 Å, then the wavelength of the corresponding anti-Stokes line is **[B.H.U-2011]**

(a) 5400 Å (b) 5490 Å (c) 5580 Å (d) 5550 Å

**Soln.** The wave number of anti-Stokes line is given by

$$\bar{v}_{\text{anti-Stokes}} - \bar{v}_{\text{Raman}} = \bar{v}_{\text{Raman}} - \bar{v}_{\text{Stokes}}$$

$$\Rightarrow \bar{v}_{\text{antiStokes}} = 2\bar{v}_{\text{Raman}} - \bar{v}_{\text{Raman}} - \bar{v}_{\text{Stokes}}$$

$$= 2\times\frac{1}{5460} - \frac{1}{5520}$$

$$= \frac{2\times 5520 - 5400}{5460\times 5520} = 0.0001851\,\text{Å}^{-1}$$

$\therefore$ Wavelength of anti-Stokes line $= \dfrac{1}{\bar{v}_{\text{anti-Stokes}}} = \dfrac{1}{0.0001851}\,\text{Å} \approx 5400\,\text{Å}$

**Correct option is (a)**

12. Which of the following transitions is allowed ? **[B.H.U-2010]**

(a) ${}^2S_{1/2} \to {}^2D_{3/2}$ (b) ${}^2P_{1/2} \to {}^2D_{3/2}$ (c) ${}^2S_{1/2} \to {}^2S_{1/2}$ (d) ${}^2D_{5/2} \to {}^2P_{1/2}$

**Soln.** Transition selection rule

$\Delta S = 0$

$\Delta L = 0, \pm 1$

$\Delta J = 0, \pm 1$ but $J = 0 \not\leftrightarrow J = 0$

$${}^2S_{1/2} \longrightarrow {}^2D_{3/2}$$

$S = \frac{1}{2}$ $\quad S = \frac{1}{2} \Rightarrow \Delta S = 0$

$J = \frac{1}{2}$ $\quad J = \frac{1}{2} \Rightarrow \Delta J = 0$

$L = 0$ $\quad L = 2 \Rightarrow \Delta L = 2$ (not allowed)

$\therefore$ Option (a) is incorrect

$${}^2P_{1/2} \longrightarrow {}^2D_{3/2}$$

$S=\frac{1}{2} \qquad S=\frac{1}{2}\Rightarrow \Delta S=0$

$J=\frac{1}{2} \qquad J=\frac{3}{2}\Rightarrow \Delta J=1$

$L=1 \qquad L=2\Rightarrow \Delta L=1$

$\therefore$ Option (b) is correct

${}^2S_{1/2}\longrightarrow {}^2S_{1/2}$

Some state transition not allowed

$\therefore$ Option (c) is incorrect

${}^2D_{5/2}\longrightarrow {}^2P_{1/2}$

$S=\frac{1}{2} \qquad S=\frac{1}{2}\Rightarrow \Delta S=0$

$J=\frac{5}{2} \qquad J=\frac{1}{2}\Rightarrow \Delta J=2$ (not allowed)

$L=2 \qquad L=1\Rightarrow \Delta L=1$

**Correct option is (b)**

13. An atom of magnetic quantum number $m_l$ and mass m is placed in a magnetic field B, then magnetic energy is **[B.H.U-2011]**

(a) $m_\ell\left(\frac{e\hbar}{2m}\right)B^2$ (b) $m\left(\frac{e\hbar}{2m_\ell}\right)B$ (c) $m_\ell\left(\frac{e\hbar}{2m}\right)B$ (d) $m\left(\frac{e\hbar}{2m_\ell}\right)B^2$

**Soln.** The magnetic potential energy of an atom in a magnetic field is given by

$E=-\vec{\mu}_L\cdot\vec{B}=-\mu_{L_z}B_z$ (Assuming $B$ in $z$-direction)

Where $\mu_L=-\frac{e}{2m}m_l\hbar$ (Orbital magnetic moment)

$$E=-\left(-\frac{e}{2m}m_l\hbar\right)B_z=m_l\left(\frac{e\hbar}{2m}\right)B$$

**Correct option is (c)**

14. A sample of certain element is placed in a 0.300 tesla magnetic field and suitably excited. How far apart are the Zeeman components of the 450 nm spectral line of this element? **[B.H.U-2011]**

(a) $5.66\times10^{-12}$ metre (b) $2.83\times10^{-12}$ metre (c) 0.0014 nm (d) 0.0042 nm

**Soln.** Given : $|\vec{B}|=0.3T,\ \lambda=450\ nm$

$$\Delta E=\left(\frac{e\hbar}{2m}\right)B=h\,\Delta f=h\Delta\left(\frac{c}{\lambda}\right)=-hc\frac{\Delta\lambda}{\lambda^2}$$

$$\Rightarrow |\Delta\lambda|=\left(\frac{e\hbar}{2m}\right)B\frac{\lambda^2}{hc}=\frac{eB\lambda^2}{4\pi\,mc}$$

$$|\Delta\lambda| = \frac{(1.6\times10^{-19})\times(0.3)\times(450\times10^{-9})^2}{4\pi\times(9.11\times10^{-3})\times(3\times10^{8})} = 0.00203\text{ nm} = 2.83\times10^{-12}\text{ m}$$

**Correct option is (a)**

15. The total number of terms arising due to one $4p$ and one $4d$ electrons i.e., outer configuration of an atom as $4p$ $4d$ is

(a) 6 (b) 12 (c) 9 (d) 3 **[B.H.U-2015]**

**Soln.** For $4p\,4d$:

For this, $s_1 = \frac{1}{2},\ s_2 = \frac{1}{2},\ \ell_1 = 1,\ \ell_2 = 2$

Therefore, $s = |s_1 - s_2|, |s_1 - s_2| + 1 = 0, 1$

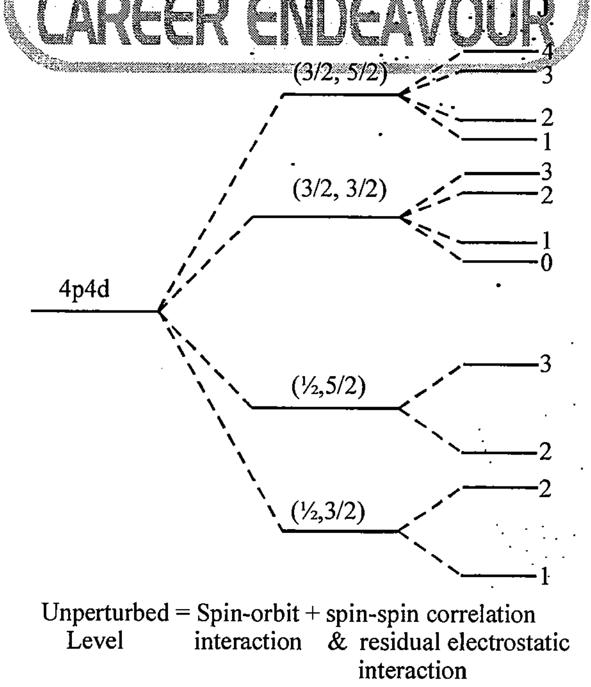
Multiplicity $(2s+1) = 1, 3$ (single, triplet)

$L = |\ell_1 - \ell_2|, |\ell_1 - \ell_2| + 1, \ldots = 1, 2, 3$ ($P, D, F$ states)

Since, $J = |L - S|, \ldots, |t - s|$

| Singlet terms ($s = 0$) | Triplet terms ($s = 1$) |
|---|---|
| $\left.\begin{matrix}S=0\\L=1\end{matrix}\right\}$ $J = 1,\ {}^1P_1$ | $\left.\begin{matrix}S=1\\L=1\end{matrix}\right\}$ $J = 0,1,2;\ {}^3P_{0,1,2}$ |
| $\left.\begin{matrix}S=0\\L=2\end{matrix}\right\}$ $J = 2,\ {}^1D_2$ | $\left.\begin{matrix}S=1\\L=2\end{matrix}\right\}$ $J = 1,2,3;\ {}^3D_{1,2,3}$ |
| $\left.\begin{matrix}S=0\\L=3\end{matrix}\right\}$ $J = 3,\ {}^1F_3$ | $\left.\begin{matrix}S=1\\L=3\end{matrix}\right\}$ $J = 2,3,4;\ {}^3F_{2,3,4}$ |

Thus a single degenerate level of configuration $4p\,4d$ is splitted into 12 levels as shown below.


J
(3/2, 5/2)
4
3
2
1
(3/2, 3/2)
3
2
1
0
4p4d
(½,5/2)
3
2
(½,3/2)
2
1
Unperturbed Level
= Spin-orbit interaction
+ spin-spin correlation & residual electrostatic interaction

# *LIST OF REFERENCES*

1. Concepts of Modern Physics (6th Edition): **Aurther Beiser**
2. Quantum Physics: **H. C. Verma**
3. Quantum Mechanics, Concepts and Applications: **Nouredine Zettili**
4. Quantum Mechanics: **G. Aruldhas**
5. Quantum Mechanics, 500 problems with solutions: **G. Aruldhas**
6. Introduction to Quantum Mechanics: **David J. Griffiths**
7. Nuclear Physics: **D. C. Tayal**
8. Nuclear Physics: **S. N. Ghosal**
9. Introductory Nuclear Physics: **Kenneth S. Krane**
10. Atomic & Molecular Spectra, Laser: **Raj Kumar**

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